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Abstract: With the help of two Skorokhod embeddings, we construct martingales which enjoy the Brownian scaling property and the (inhomogeneous) Markov property. The second method necessitates randomization, but allows to reach any law with finite moment of order 1, centered, as the distribution of such a martingale at unit time. The first method does not necessitate randomization, but an additional restriction on the distribution at unit time is needed.

Key words: Skorokhod embeddings, Hardy-Littlewood functions, convex order, Schauder fixed point theorem, self-similar martingales, Karamata's representation theorem.

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General introduction

I.1 Our general program
This work consists of two parts A and B which both have the same purpose, i.e.: to construct a large class of martingales $(M_t, t \geq 0)$ which satisfy the two additional properties:

(a) $(M_t, t \geq 0)$ enjoys the Brownian scaling property:

$$\forall c > 0, \quad (M_{c^2 t}, t \geq 0) \overset{\text{(law)}}{=} (cM_t, t \geq 0)$$

(b) $(M_t, t \geq 0)$ is (inhomogeneous) Markovian.

The paper by Madan and Yor [MY02] developed three quite different methods to achieve this aim. In the following Parts A and B, we further develop two different Skorokhod embedding methods for the same purpose. In the end, the family of laws $\mu \sim M_1$ which are reached in Part A is notably bigger than in [MY02], while the method in Part B allows to reach any centered probability measure μ (with finite moment of order 1).

I.2 General facts about Skorokhod embeddings
For ease of the reader, we recall briefly the following facts:

• Consider a real valued, integrable and centered random variable X . Realizing a *Skorokhod embedding* of X into the Brownian motion B, consists in constructing a stopping time τ such that:

(Sk1) $B_\tau \overset{\text{(law)}}{=} X$

(Sk2) $(B_{u\wedge\tau}\ ,\ u \geq 0)$ is a uniformly integrable martingale.

There are many ways to realize such a Skorokhod embedding. J. Oblój ([Obl04]) numbered twenty one methods scattered in the literature. These methods separate (at least) in two kinds:

- the time τ is a stopping time relative to the natural filtration of the Brownian motion B;
- the time τ is a stopping time relative to an enlargement of the natural filtration of the Brownian motion, by addition of extra random variables, independent of B.

In the second case, the stopping time τ is called a *randomized* stopping time. We call the corresponding embedding a *randomized Skorokhod embedding.*

• Suppose that, for every $t \geq 0$, there exists a stopping time τ_t satisfying **(Sk1)** and **(Sk2)** with $\sqrt{t}\,X$ replacing X. If *the family of stopping* *times* $(\tau_t\ ,\ t \geq 0)$ *is a.s. increasing*, then the process $(B_{\tau_t}\ ,\ t \geq 0)$ is a martingale and, for every fixed $t \geq 0$ and for every $c > 0$,

$$B_{\tau_{c^2 t}} \overset{\text{(law)}}{=} c\sqrt{t}\,X \overset{\text{(law)}}{=} c\,B_{\tau_t}\ ,$$

which, a priori, is a weaker property than the scaling property **(a)**. Nevertheless, the process $(B_{\tau_t}\ ,\ t \geq 0)$ appears to be a good candidate to satisfy **(a)**, **(b)** and $B_{\tau_1} \overset{\text{(law)}}{=} X$.

• Part A consists in using the Azéma-Yor algorithm, which yields a Skorokhod embedding of the first kind, whereas Part B hinges on a Skorokhod embedding of the second kind, both in order to obtain martingales $(B_{\tau_t}, t \geq 0)$ which satisfy **(a)** and **(b)**.
Of course at the beginning of each part, we shall give more details, pertaining to the corresponding embedding, so that Parts A and B may be read independently.

I.3 Examples of such martingales
The most famous examples of martingales satisfying **(a)** and **(b)** are, without any contest, Brownian motion $(B_t, t \geq 0)$ and the Azéma martingale $(\xi_t := \mathrm{sgn}(B_t)\sqrt{t - g_t},\ t \geq 0)$ where $g_t := \sup\{s \leq t; B_s = 0\}$.
The study of the latter martingale $(\xi_t, t \geq 0)$, originally discovered by Azéma [Azé85], was then developed by Emery [Éme89, Éme96], Azéma-Yor [AY89], Meyer [Mey89a]. In particular, M. Emery established that Azéma martingale enjoys the Chaotic Representation Property (CRP). This discovery and subsequent studies were quite spectacular because, until then, it was commonly believed that the only two martingales which enjoy the CRP were Brownian motion and the compensated Poisson process. In fact, it turns out that a number of other martingales enjoying the CRP, together with **(a)** and **(b)**, could be constructed, and were the subject of studies by P.A Meyer [Mey89b], M. Emery [Éme96], M. Yor [Yor97, Chapter 15]. The structure equation concept played quite an important role there. However, we shall not go further into this topic, which lies outside the scope of the present paper.

I.4 Relations with peacocks
Since X is an integrable and centered r.v., the process $(\sqrt{t}X, t \geq 0)$ is increasing in the convex order (see [HPRY]). We call it a peacock. It is known from Kellerer [Kel72] that to any peacock $(\Pi_t, t \geq 0)$, one can associate a (Markovian) martingale $(M_t, t \geq 0)$ such that, for any fixed $t \geq 0$, $M_t \overset{\text{(law)}}{=} \Pi_t$, i.e.: $(M_t, t \geq 0)$ and $(\Pi_t, t \geq 0)$ have the same one-dimensional marginals. Given a peacock, it is generally difficult to exhibit an associated martingale. However, in the particular case $\Pi_t = \sqrt{t}X$ which we consider here, the process $(B_{\tau_t}\ ,\ t \geq 0)$ presented above provides

us with an associated martingale.

I.5 A warning

It may be tempting to think that the whole distribution of a martingale $(M_t, t \geq 0)$ which satisfies **(a)** (and **(b)**) is determined by the law of M_1. This is quite far from being the case, as a number of recent papers shows; the interested reader may look at Albin [Alb08], Oleszkiewicz [Ole08], Hamza-Klebaner [HK07]... We thank David Baker [Bak09] and David Hobson [Hob09] for pointing out, independently, these papers to us.

Part A

Construction via the Skorokhod embedding of Azéma-Yor

A.1 Introduction

A.1.1 Program

The methodology developed in this part is AYUS (=Azéma-Yor Under Scaling), following the terminology in [MY02]. Precisely, given a r.v. X with probability law μ, we shall use the Azéma-Yor embedding algorithm simultaneously for all distributions μ_t indexed by $t \geq 0$ where:

$$\forall t \geq 0, \quad \mu_t \sim \sqrt{t}X. \tag{A.1.1}$$

More precisely, if $(B_t, t \geq 0)$ denotes a Brownian motion and $(S_t := \sup\limits_{u \leq t} B_u, t \geq 0)$, we seek probability measures μ such that the family of stopping times:

$$T_{\mu_t} := \inf\{u \geq 0; S_u \geq \psi_{\mu_t}(B_u)\}$$

where

$$\psi_{\mu_t}(x) = \frac{1}{\mu_t([x, +\infty[)} \int_{[x,+\infty[} y\mu_t(dy)$$

increases, or equivalently, the family of functions $\left(\psi_{\mu_t}(x) = \sqrt{t}\psi_\mu\left(\frac{x}{\sqrt{t}}\right)\right)_{t \geq 0}$ increases (pointwise in x). (Since $\mu = \mu_1$, we write ψ_μ for ψ_{μ_1}). This program was already started in Madan-Yor, who came up with the (easy to prove) necessary and sufficient condition on μ:

$$a \longmapsto D_\mu(a) := \frac{a}{\psi_\mu(a)} \text{ is increasing on } \mathbb{R}_+. \tag{$M{\cdot}Y$}$$

Our main contribution in this Part A is to look for nice, easy to verify, sufficient conditions on μ which ensure that $(M{\cdot}Y)$ is satisfied. Such a condition

has been given in [MY02] (Theorems 4 and 5). In the following part A:
- We discuss further this result of Theorem 4 by giving equivalent conditions for it; this study has a strong likeness with (but differs from) Karamata's representation theorem for slowly varying functions (see, e.g. Bingham-Goldie-Teugels [BGT89, Chapter 1, Theorems 1.3.1 and 1.4.1])
- Moreover, we also find different sufficient conditions for $(M \cdot Y)$ to be satisfied.
With the help of either of these conditions, it turns out that many subprobabilities μ on $\mathbb{R}_+$ satisfy $(M \cdot Y)$; in particular, all beta and gamma laws satisfy $(M \cdot Y)$.

A.1.2 A forefather

A forefather of the present paper is Meziane-Yen-Yor [MYY09], where a martingale $(M_t, t \geq 0)$ which enjoys **(a)** and **(b)** and is distributed at time 1 as $\varepsilon\sqrt{g}$ with ε a Bernoulli r.v. and g an independent arcsine r.v. was constructed with the same method. Thus, the martingale $(M_t, t \geq 0)$ has the same one-dimensional marginals as Azéma's martingale $(\xi_t, t \geq 0)$ presented in **I.3** although the laws of M and ξ differ. Likewise in [MY02], Madan and Yor construct a purely discontinuous martingale $(N_t, t \geq 0)$ which enjoys **(a)** and **(b)** and has the same one-dimensional marginals as a Brownian motion $(B_t, t \geq 0)$.

A.1.3 Plan

The remainder of this part is organised as follows: Sections A.2 to A.4 deal with the case of measures μ with support in $]-\infty, 1]$, 1 belonging to the support of μ, while Section A.5 deals with a generic measure μ whose support is $\mathbb{R}$. More precisely:
• First, Section A.2 consists in recalling the Azéma-Yor algorithm and the Madan-Yor condition $(M \cdot Y)$, and then presenting a number of important quantities associated with μ, whether or not $(M \cdot Y)$ is satisfied. Elementary relations between these quantities are established, which will ease up our discussion later on.

• Section A.3: when $(M \cdot Y)$ is satisfied, it is clear that there exists a subprobability ν_μ on $]0, 1[$ such that:

$$D_\mu(a) = \nu_\mu(]0, a[), \quad a \in [0, 1]. \tag{A.1.2}$$

We obtain relations between quantities relative to μ and ν_μ.
In particular:
- In Subsection A.3.2, we establish a one-to-one correspondence between two sets of probabilities μ and ν.

- Subsection A.3.3 consists in the study in the particular case of $(M\cdot Y)$ when:

$$\frac{a}{\psi_\mu(a)} = \frac{1}{Z}\int_0^{+\infty} (1-e^{-ax})\rho(dx)$$

for certain positive measures ρ, where $Z = \int_0^{+\infty} (1-e^{-x})\rho(dx)$ is the normalizing constant which makes: $D_\mu(a) = \frac{a}{\psi_\mu(a)}$ a distribution function on $[0,1]$.

- Subsection A.3.4 gives another formulation of this correspondence.

• Section A.4 consists in the presentation of a number of conditions $(S_0)-(S_5)$ and subconditions (S_i') which suffice for the validity of $(M\cdot Y)$.

• Section A.5 tackles the case of a measure μ whose support is $\mathbb{R}$, and gives a sufficient condition for the existence of a probability ν_μ which satisfies (A.1.2).

• Finally, in Section A.6, many particular laws μ are illustrated in the form of graphs. We also give an example where $(M\cdot Y)$ is not satisfied, which, given the preceding studies, seems to be rather the exception than the rule.

A.2 General overview of this method

A.2.1 The Azéma-Yor algorithm for Skorokhod embedding

We start by briefly recalling the Azéma-Yor algorithm for Skorokhod embedding. Let μ be a probability on $\mathbb{R}$ such that:

$$\int_{-\infty}^{+\infty} |x|\mu(dx) < \infty \quad \text{and} \quad \int_{-\infty}^{+\infty} x\mu(dx) = 0. \tag{A.2.1}$$

We define its Hardy-Littlewood function ψ_μ by:

$$\psi_\mu(x) = \frac{1}{\mu([x,+\infty[)}\int_{[x,+\infty[} y\mu(dy).$$

In the case where there exists $x \geq 0$ such that $\mu([x,+\infty[) = 0$, we set $\alpha = \inf\{x \geq 0; \mu([x,+\infty[) = 0\}$ and $\psi_\mu(x) = \alpha$ for $x \geq \alpha$. Let $(B_t, t \geq 0)$ be a standard Brownian motion. Azéma-Yor [AY79] introduced the stopping time:

$$T_\mu := \inf\{t \geq 0; S_t \geq \psi_\mu(B_t)\}$$

where $S_t := \sup_{s\leq t} B_s$ and showed:

Theorem A.2.1 (Azéma-Yor, [AY79]).
1) $(B_{t\wedge T_\mu}, t \geq 0)$ *is a uniformly integrable martingale.*
2) *The law of* B_{T_μ} *is* μ: $B_{T_\mu} \sim \mu$.

To prove Theorem A.2.1, Azéma-Yor make use of the martingales:

$$\left(\varphi(S_t)(S_t - B_t) + \int_{S_t}^{+\infty} dx\varphi(x),\ t \geq 0\right)$$

for any $\varphi \in L^1(\mathbb{R}_+, dx)$. Rogers [Rog81] shows how to derive Theorem A.2.1 from excursion theory, while Jeulin-Yor [JY81] develop a number of results about the laws of $\displaystyle\int_0^{T_\mu} h(B_s)ds$ for a generic function h.

A.2.2 A result of Madan-Yor

Madan-Yor [MY02] have exploited this construction to find martingales $(X_t, t \geq 0)$ which satisfy **(a)** and **(b)**. More precisely:

Proposition A.2.2.
Let X *be an integrable and centered r.v. with law* μ. *Let, for every* $t \geq 0$, $\widetilde{X}_t := \sqrt{t}X$. *Denote by* μ_t *the law of* $\widetilde{X}_t$ *and by* ψ_t $(= \psi_{\mu_t})$ *the Hardy-Littlewood function associated to* μ_t:

$$\psi_t(x) = \frac{1}{\mu_t([x, +\infty[)} \int_{[x,+\infty[} y\mu_t(dy) = \sqrt{t}\psi_1\left(\frac{x}{\sqrt{t}}\right)$$

and by $T_t^{(\mu)}$ *the Azéma-Yor stopping time (which we shall also denote* T_{ψ_t}*):*

$$T_t^{(\mu)} := \inf\{u \geq 0; S_u \geq \psi_t(B_u)\} \tag{A.2.2}$$

(Theorem A.2.1 asserts that $B_{T_t^{(\mu)}} \sim \mu_t$*). We assume furthermore:*

$$t \longmapsto T_t^{(\mu)} \text{ is a.s. increasing} \tag{I}$$

Then:
1) *The process* $\left(X_t^{(\mu)} := B_{T_t^{(\mu)}}, t \geq 0\right)$ *is a martingale, and an (inhomogeneous) Markov process.*
2) *The process* $\left(X_t^{(\mu)}, t \geq 0\right)$ *enjoys the Brownian scaling property, i.e for every* $c > 0$:

$$\left(X_{c^2t}^{(\mu)}, t \geq 0\right) \overset{(law)}{=} \left(cX_t^{(\mu)}, t \geq 0\right)$$

In particular, $\left(X_t^{(\mu)} := B_{T_t^{(\mu)}}, t \geq 0\right)$ *is a martingale associated to the peacock* $(\sqrt{t}X, t \geq 0)$ *(see Introduction).*

Proof of Proposition A.2.2
Point 1) is clear. See in particular [MY02] where the infinitesimal generator of $(X_t^{(\mu)}, t \geq 0)$ is computed. It is therefore sufficient to prove Point 2). Let $c > 0$ be fixed.
$i)$ From the scaling property of Brownian motion:

$$(S_{c^2t}, B_{c^2t},\ t \geq 0) \overset{(\text{law})}{=} (cS_t, cB_t,\ t \geq 0)\,,$$

and the definition (A.2.2) of T_{ψ_t}, we deduce that:

$$\left(B_{T_{\psi_t}}, t \geq 0\right) \overset{(\text{law})}{=} \left(cB_{T_{\psi_t^{(c)}}}, t \geq 0\right) \tag{A.2.3}$$

with $\psi_t^{(c)}(x) := \dfrac{1}{c}\psi_t(cx)$.
$ii)$ An elementary computation yields:

$$\psi_t(x) = \sqrt{t}\psi\left(\frac{x}{\sqrt{t}}\right) \tag{A.2.4}$$

with $\psi := \psi_1 = \psi_\mu$. We obtain from (A.2.4) that:

$$\psi_{c^2t}^{(c)}(x) = \frac{1}{c}\psi_{c^2t}(cx) = \sqrt{t}\psi\left(\frac{x}{\sqrt{t}}\right) = \psi_t(x). \tag{A.2.5}$$

Finally, gathering $i)$ and $ii)$, it holds:

$$\begin{aligned}\left(X_{c^2t}^{(\mu)} := B_{T_{\psi_{c^2t}}}, t \geq 0\right) &\overset{(\text{law})}{=} \left(cB_{T_{\psi_{c^2t}^{(c)}}}, t \geq 0\right) \quad \text{(from (A.2.3))}\\ &\overset{(\text{law})}{=} \left(cB_{T_{\psi_t}} = cX_t^{(\mu)}, t \geq 0\right) \quad \text{(from (A.2.5))}\end{aligned}$$

□

A.2.2.1 Examples

In the paper [MYY09] which is a forefather of the present paper, the following examples were studied in details:

$i)$ The dam-drawdown example:

$$\mu_t(dx) = \frac{1}{t}\exp\left(-\frac{1}{t}(x+t)\right) 1_{[-t,+\infty[}(x)dx$$

which yields to the stopping time $T_t := \inf\{u \geq 0; S_u - B_u = t\}$. Recall that from Lévy's theorem, $(S_u - B_u, u \geq 0)$ is a reflected Brownian motion.

ii) The "BES(3)-Pitman" example:

$$\mu_t(dx) = \frac{1}{2t} 1_{[-t,t]}(x)dx$$

which corresponds to the stopping time $T_t := \inf\{u \geq 0; 2S_u - B_u = t\}$. Recall that from Pitman's Theorem, $(2S_u - B_u, u \geq 0)$ is distributed as a Bessel process of dimension 3 started from 0.

iii) The Azéma-Yor "fan", which is a generalization of the two previous examples:

$$\mu^{(\alpha)}(dx) = \frac{\alpha}{t}\left(\alpha - \frac{(1-\alpha)x}{t}\right)^{\frac{2\alpha-1}{1-\alpha}} 1_{[-t,\frac{\alpha t}{1-\alpha}]}(x)dx, \qquad (0 < \alpha < 1)$$

which yields to the stopping time $T_t^{(\alpha)} := \inf\{u \geq 0; S_u = \alpha(B_u + t)\}$. Example *i*) is obtained by letting $\alpha \to 1^-$.

A.2.2.2 The $(M{\cdot}Y)$ condition

Proposition A.2.2 highlights the importance of condition (I) (see Proposition A.2.2 above) for our search of martingales satisfying conditions **(a)** and **(b)**. We now wish to be able to read "directly" from the measure μ whether (I) is satisfied or not. The answer to this question is presented in the following Lemma

Lemma A.2.3 (Madan-Yor [MY02], Lemma 3)**.**
Let $X \sim \mu$ satisfy (A.2.1). We define:

$$D_\mu(x) := \frac{x\overline{\mu}(x)}{\int_{[x,+\infty[} y\mu(dy)} \quad \textit{with } \overline{\mu}(x) := \mathbb{P}(X \geq x) = \int_{[x,+\infty[} \mu(dy)$$

Then (I) is satisfied if and only if :

$$x \longmapsto D_\mu(x) \textit{ is increasing on } \mathbb{R}_+. \qquad (M{\cdot}Y)$$

Proof of Lemma A.2.3
Condition (I) is equivalent to the increase, for any given $x \in \mathbb{R}$, of the function $t \longmapsto \psi_t(x)$. From (A.2.4), $\psi_t(x) = \sqrt{t}\psi\left(\frac{x}{\sqrt{t}}\right)$, hence, if $x \leq 0$, since ψ is a positive and increasing function, $t \longmapsto \psi_t(x)$ is increasing. For $x > 0$, we set $a_t = \frac{x}{\sqrt{t}}$; thus:

$$\psi_t(x) = \sqrt{t}\psi\left(\frac{x}{\sqrt{t}}\right) = x\frac{\psi(a_t)}{a_t},$$

and, $t \longmapsto a_t$ being a decreasing function of t, condition (I) is equivalent to the increase of the function $a \longmapsto \dfrac{a}{\psi(a)} = D_\mu(a)$.

□

A remarkable feature of this result is that (I) only depends on the restriction of μ to $\mathbb{R}_+$. This is inherited from the asymmetric character of the Azéma-Yor construction in which $\mathbb{R}_+$, via $(S_u, u \geq 0)$ plays a special role.
Now, let $\widetilde{\mu}$ be a probability on $\mathbb{R}$ satisfying (A.2.1). Since our aim is to obtain conditions equivalent to $(M{\cdot}Y)$, i.e.:

$$x \longmapsto D_{\widetilde{\mu}}(x) := \frac{x \int_{[x,+\infty[} \widetilde{\mu}(dy)}{\int_{[x,+\infty[} y\widetilde{\mu}(dy)} \text{ increases on } \mathbb{R}_+, \tag{A.2.6}$$

it suffices to study $D_{\widetilde{\mu}}$ on $\mathbb{R}_+$. Clearly, this function (on $\mathbb{R}_+$) depends only on the restriction of $\widetilde{\mu}$ to $\mathbb{R}_+$, which we denote by μ. Observe that $(M{\cdot}Y)$ remains unchanged if we replace μ by $\lambda\mu$ where λ is a positive constant.
Besides, we shall restrict our study to the case where $\widetilde{\mu}$ is carried by $]-\infty, k]$, i.e. where $\mu = \widetilde{\mu}_{|\mathbb{R}_+}$ is a subprobability on $[0, k]$. To simplify further, but without loss of generality, we shall take $k = 1$ and assume that 1 belongs to the support of μ. In Section A.5, we shall study briefly the case where μ is a measure whose support is $\mathbb{R}_+$.

A.2.3 Notation

In this Subsection, we present some notation which shall be in force throughout the remainder of the paper. Let μ be a positive measure on $[0, 1]$, with finite total mass, and whose support contains 1. We denote by $\overline{\mu}$ and $\overline{\overline{\mu}}$, respectively its tail and its double tail functions:

$$\overline{\mu}(x) = \int_{[x,1]} \mu(dy) = \mu([x, 1]) \quad \text{ and } \quad \overline{\overline{\mu}}(x) = \int_x^1 \overline{\mu}(y)dy.$$

Note that $\overline{\mu}$ is left-continuous, $\overline{\overline{\mu}}$ is continuous, and $\overline{\mu}$ and $\overline{\overline{\mu}}$ are both decreasing functions. Furthermore, it is not difficult to see that a function $\Lambda : [0, 1] \longrightarrow \mathbb{R}_+$ is the double tail function of a positive finite measure on $[0, 1]$ if and only if Λ is a convex function on $[0, 1]$, left-differentiable at 1, right-differentiable at 0, and satisfying $\Lambda(1) = 0$.
We also define the tails ratio u_μ associated to μ:

$$u_\mu(x) = \overline{\mu}(x)/\overline{\overline{\mu}}(x), \qquad x \in [0, 1[.$$

Here is now a lemma of general interest which bears upon positive measures:

Lemma A.2.4 (Pierre [Pie80] or Revuz-Yor [RY99], Chapter VI Lemma 5.1)**.**
1) *For every* $x \in [0, 1[$*:*

$$\overline{\mu}(x) = \overline{\overline{\mu}}(0) u_\mu(x) \exp\left(-\int_0^x u_\mu(y)dy\right) \tag{A.2.7}$$

and u_μ is left-continuous.
2) *Let $v : [0,1[\longrightarrow \mathbb{R}_+$ be a left-continuous function such that, for all $x \in [0,1[$:*

$$\overline{\mu}(x) = \overline{\overline{\mu}}(0)v(x)\exp\left(-\int_0^x v(y)dy\right)$$

Then, $v = u_\mu$.

Proof of Lemma A.2.4
1) We first prove Point 1). For $x \in [0,1[$, we have:

$$-\int_0^x \frac{\overline{\mu}(y)}{\overline{\overline{\mu}}(y)}dy = \int_0^x \frac{d\overline{\overline{\mu}}(y)}{\overline{\overline{\mu}}(y)} = \left[\log\overline{\overline{\mu}}(y)\right]_0^x = \log\overline{\overline{\mu}}(x) - \log\overline{\overline{\mu}}(0),$$

hence,

$$\overline{\overline{\mu}}(0)u_\mu(x)\exp\left(-\int_0^x u_\mu(y)dy\right) = \overline{\overline{\mu}}(0)\frac{\overline{\mu}(x)}{\overline{\overline{\mu}}(x)}\exp\left(\log\frac{\overline{\overline{\mu}}(x)}{\overline{\overline{\mu}}(0)}\right) = \overline{\mu}(x).$$

2) We now prove Point 2). Let $U_\mu(x) := \int_0^x u_\mu(y)dy$ and $V(x) := \int_0^x v(y)dy$. Relation (A.2.7) implies:

$$\begin{aligned} u_\mu(x)\exp\left(-U_\mu(x)\right) &= v(x)\exp\left(-V(x)\right), \quad \text{i.e.} \\ \left(\exp\left(-U_\mu(x)\right)\right)' &= \left(\exp\left(-V(x)\right)\right)', \quad \text{hence} \\ \exp\left(-U_\mu(x)\right) &= \exp\left(-V(x)\right) + c. \end{aligned}$$

(The above derivatives actually denote left-derivatives). Now, since, $U_\mu(0) = V(0) = 0$, we obtain $c = 0$ and $U_\mu = V$. Then, differentiating, and using the fact that u_μ and v are left-continuous, we obtain: $u_\mu = v$.

□

Remark A.2.5.
Since $\overline{\mu}$ is a decreasing function, we see, by differentiating (A.2.7), that the function u_μ satisfies:
- if $\overline{\mu}$ is differentiable, then so is u_μ and $u'_\mu \le u^2_\mu$,
- more generally, the distribution on$]0,1[$: $u^2_\mu - u'_\mu$, is a positive measure.

Note that if $\mu(dx) = h(x)dx$, then:

$$u^2_\mu - u'_\mu = \left(\overline{\mu}/\overline{\overline{\mu}}\right)^2 - \left(\frac{\overline{\mu}^2 - h\overline{\overline{\mu}}}{\overline{\overline{\mu}}^2}\right) = h/\overline{\overline{\mu}} \ge 0.$$

By (A.2.7), we have, for any $x \in [0,1[$,

$$\overline{\overline{\mu}}(x) = \overline{\overline{\mu}}(0)\exp\left(-\int_0^x u_\mu(y)dy\right).$$

Since $\overline{\overline{\mu}}(0) = \displaystyle\int_{[0,1]} y\mu(dy) > 0$ and $\overline{\overline{\mu}}(1) = 0$, we obtain:

$$\forall x < 1, \int_0^x u_\mu(y)dy < \infty \quad \text{and} \quad \int^{1^-} u_\mu(y)dy = +\infty \tag{A.2.8}$$

As in Subsection A.2.1, we now define the Hardy-Littlewood function ψ_μ associated to μ:

$$\begin{cases} \psi_\mu(a) = \dfrac{1}{\mu([a,1])} \displaystyle\int_{[a,1]} y\mu(dy) \quad , & a \in [0,1[\\ \psi_\mu(1) = 1 \end{cases}$$

and the Madan-Yor function associated to μ:

$$D_\mu(a) = \frac{a}{\psi_\mu(a)}, \qquad a \in [0,1].$$

In particular, $D_\mu(1) = 1$ and $D_\mu(0) = 0$. Note that, integrating by parts:

$$\overline{\overline{\mu}}(a) = \int_a^1 (y-a)\mu(dy) = \overline{\mu}(a)\left(\psi_\mu(a) - a\right),$$

hence, $u_\mu(a) = \dfrac{1}{\psi_\mu(a) - a}$ and, consequently:

$$D_\mu(a) = \frac{a}{\psi_\mu(a) - a + a} = \frac{a}{(1/u_\mu(a)) + a} = \frac{au_\mu(a)}{au_\mu(a) + 1}. \tag{A.2.9}$$

We sum up all the previous notation in a Table, for future references:

$\mu(dx)$	A finite positive measure on $[0,1]$ whose support contains 1.
$\overline{\mu}(a) = \mu([a,1])$	Tail function associated to μ
$\overline{\overline{\mu}}(a) = \int_a^1 \overline{\mu}(x)dx$	Double tail function associated to μ
$u_\mu(a) = \overline{\mu}(a)/\overline{\overline{\mu}}(a) = \dfrac{1}{\psi_\mu(a) - a}$	Tails ratio function associated to μ
$\psi_\mu(a) = \dfrac{1}{\overline{\mu}(a)} \int_{[a,1]} x\mu(dx)$	Hardy-Littlewood function associated to μ
$D_\mu(a) = \dfrac{a}{\psi_\mu(a)} = \dfrac{au_\mu(a)}{au_\mu(a)+1}$	Madan-Yor function associated to μ

A.3 Some conditions which are equivalent to $(M{\cdot}Y)$

A.3.1 A condition which is equivalent to $(M{\cdot}Y)$

Let μ denote a positive measure on $[0,1]$, with finite total mass, and whose support contains 1. We now study the condition $(M{\cdot}Y)$ in more details.

A.3.1.1 Elementary properties of D_μ

i) From the obvious inequalities, for $x \in [0,1]$:

$$x\overline{\mu}(x) = x\int_{[x,1]} \mu(dy) \leq \int_{[x,1]} y\mu(dy) \leq \int_{[x,1]} \mu(dy) = \overline{\mu}(x)$$

we deduce that ψ_μ and D_μ are left-continuous on $]0,1]$, and for every $x \in [0,1]$,

$$x \leq \psi_\mu(x) \leq 1 \quad \text{and} \quad x \leq D_\mu(x) \leq 1. \tag{A.3.1}$$

ii) We now assume that μ admits a density h; then:

- if h is continuous at 0, then: $D'_\mu(0^+) = \overline{\mu}(0)/\overline{\overline{\mu}}(0)$,
- if h is continuous at 1, and $h(1) > 0$, then: $D'_\mu(1^-) = \frac{1}{2}$,
- if h admits, in a neighbourhood of 1, the equivalent: $h(1-x) \underset{x\to 0}{=} Cx^\alpha + o(x^\alpha)$, with $C, \alpha > 0$ then: $D'_\mu(1^-) = \dfrac{1}{2+\alpha}$.

These three properties are consequences of the following formula, which holds at every point where h is continuous:

$$\frac{D'_\mu(x)}{D_\mu(x)} = \frac{1}{x} - h(x)\frac{1 - D_\mu(x)}{\overline{\mu}(x)}.$$

A.3.1.2 A condition which is equivalent to $(M{\cdot}Y)$

Theorem A.3.1. *Let μ be a finite positive measure on $[0,1]$ whose support contains 1, and u_μ its tails ratio. The following assertions are equivalent:*

i) *D_μ is increasing on $[0,1]$, i.e. $(M{\cdot}Y)$ holds.*

ii) *There exists a probability measure ν_μ on $]0,1[$ such that:*

$$\forall a \in [0,1], \quad D_\mu(a) = \nu_\mu(]0,a[). \tag{A.3.2}$$

iii) *$a \longrightarrow au_\mu(a)$ is an increasing function on $[0,1[$.*

Proof of Theorem A.3.1
Of course, the equivalence between *i*) and *ii*) holds, since $D_\mu(0) = 0$ and $D_\mu(1) = 1$. As for the equivalence between *i*) and *iii*), it follows from (A.2.9):

$$D_\mu(a) = \frac{au_\mu(a)}{au_\mu(a)+1}.$$

□

Remark A.3.2.
1) The probability measure ν_μ defined via (A.3.2) enjoys some particular properties. Indeed, from (A.3.2), it satisfies

$$\frac{\nu_\mu(]0,a[)}{a} = \frac{D_\mu(a)}{a} = \frac{1}{\psi_\mu(a)}.$$

Thus, since the function ψ_μ is increasing on $[0,1]$, the function $a \longmapsto \dfrac{\nu_\mu(]0,a[)}{a}$ is decreasing on $[0,1]$, and $\lim\limits_{a\to 0+} \dfrac{\nu_\mu(]0,a[)}{a} = \dfrac{1}{\psi_\mu(0)}$.
2) From (A.2.9), we have $\nu_\mu(]0,a[) = D_\mu(a) = \dfrac{au_\mu(a)}{au_\mu(a)+1}$, hence, for every $a \in]0,1[$:

$$u_\mu(a) = \frac{\nu_\mu(]0,a[)}{a\nu_\mu([a,1[)},$$

and, in particular, $\nu_\mu([a,1[) > 0$. Thus, with the help of (A.2.8), ν_μ necessarily satisfies the relation:

$$\int^{1^-} \frac{da}{\nu_\mu([a,1[)} = +\infty.$$

3) The function D_μ is characterized by its values on $]0,1[$ (since $D_\mu(0) = 0$ and $D_\mu(1) = 1$). Hence, D_μ only depends on the values of ψ_μ on $]0,1[$, and therefore, D_μ only depends on the restriction of μ to $]0,1]$. The value of $\mu(\{0\})$ is irrelevant for the $(M\cdot Y)$ condition.

A.3.2 Characterizing the measures ν_μ

Theorem A.3.1 invites to ask for the following question: given a probability measure ν on $]0,1[$, under which conditions on ν does there exists a positive measure μ on $[0,1]$ with finite total mass[1] such that μ satisfies $(M\cdot Y)$?
In particular, are the conditions given in Point 1) and 2) of the previous Remark A.3.2 sufficient ? In the following Theorem, we answer this question in the affirmative.

Notation. We adopt the following notation:
• $\mathcal{P}_1$ denotes the set of all probabilities μ on $[0,1]$, whose support contains 1, and which satisfy $(M\cdot Y)$.
• $\mathcal{P}_1^0 = \{\mu \in \mathcal{P}_1; \mu(\{0\}) = 0\}$.
• $\mathcal{P}_1'$ denotes the set of all probabilities ν on $]0,1[$ such that:

i) $\nu([a,1[) > 0$ for every $a \in]0,1[$,

[1] Note that since D_μ remains unchanged if we replace μ by a multiple of μ, μ can always be chosen to be a probability.

ii) $a \longmapsto \dfrac{\nu(]0,a[)}{a}$ is a decreasing function on $]0,1]$ such that

$c_\nu := \lim\limits_{a\to 0+} \dfrac{\nu(]0,a[)}{a} < \infty$,

iii) $\displaystyle\int^{1^-} \frac{da}{\nu([a,1[)} = +\infty.$

• We define a map Γ on $\mathcal{P}_1$ as follows: if $\mu \in \mathcal{P}_1$, then $\Gamma(\mu)$ is the measure ν on $]0,1[$ such that

$$D_\mu(a) = \nu(]0,a[), \quad a \in [0,1].$$

In other words, $\Gamma(\mu) = \nu_\mu$ defined by (A.3.2).
With the help of the above notation, we can state:

Theorem A.3.3.
1) $\Gamma(\mathcal{P}_1^0) = \Gamma(\mathcal{P}_1) = \mathcal{P}_1'$.
2) If $\mu \in \mathcal{P}_1$ *and* $\mu_0 \in \mathcal{P}_1^0$*, then*

$$\Gamma(\mu) = \Gamma(\mu_0) \quad \textit{if and only if} \quad \mu = \mu(\{0\})\delta_0 + (1-\mu(\{0\}))\mu_0$$

(where δ_0 *denotes the Dirac measure at 0).*
As a consequence of 1) and 2), Γ *induces a bijection between* $\mathcal{P}_1^0$ *and* $\mathcal{P}_1'$*.*

Proof of Theorem A.3.3
a) Remark A.3.2 entails that:

$$\Gamma(\mathcal{P}_1^0) \subset \Gamma(\mathcal{P}_1) \subset \mathcal{P}_1'.$$

b) We now prove $\mathcal{P}_1' \subset \Gamma(\mathcal{P}_1^0)$. Let $\nu \in \mathcal{P}_1'$. We define $u^{(\nu)}$ by:

$$\begin{cases} u^{(\nu)}(x) := \dfrac{\nu(]0,x[)}{x(1-\nu(]0,x[))} & \text{for } x \in]0,1[, \\ u^{(\nu)}(0) := c_\nu = \lim\limits_{x\to 0^+} u^{(\nu)}(x), \end{cases}$$

and we set, for $x \in [0,1[$:

$$m(x) = \frac{1}{c_\nu} u^{(\nu)}(x) \exp\left(-\int_0^x u^{(\nu)}(y)dy\right). \tag{A.3.3}$$

We remark that m is left-continuous on $]0,1[$, right-continuous at 0 and $m(0) = 1$. To prove that m is decreasing on $[0,1[$, it suffices to show that m is decreasing on $]0,1[$ or, equivalently (see Remark A.2.5), that the distribution on $]0,1[$: $\left(u^{(\nu)}\right)^2 - \left(u^{(\nu)}\right)'$, is a positive measure.
Now, from the definition of $u^{(\nu)}$, and setting:

$$\nu(a) := \nu(]0,a[),$$

we need to prove that (on $]0,1[$):

$$\begin{aligned}
&\nu^2(a)da \geq a(1-\nu(a))d\nu(a) - \nu(a)(1-\nu(a))da + a\nu(a)d\nu(a) \\
\Longleftrightarrow\ & \nu^2(a)da \geq a d\nu(a) - \nu(a)\big(1-\nu(a)\big)da \\
\Longleftrightarrow\ & 0 \geq a d\nu(a) - \nu(a)da \\
\Longleftrightarrow\ & 0 \geq d\left(\frac{\nu(a)}{a}\right).
\end{aligned}$$

The latter is ensured by Property *ii*) in the definition of $\mathcal{P}_1'$. Hence, there exists a probability μ on $[0,1]$ such that

$$\overline{\mu}(x) = m(x), \quad x \in [0,1[.$$

In particular, since m is right-continuous at 0, $\mu(\{0\}) = 0$. Using Property *iii*) in the definition of $\mathcal{P}_1'$, we obtain from (A.3.3), by integration:

$$\overline{\overline{\mu}}(0) = \frac{1}{c_\nu}.$$

Therefore, by Lemma A.2.4, $u^{(\nu)} = u_\mu$, or:

$$u_\mu(a) = \frac{\nu(]0,a[)}{a(1-\nu(]0,a[))}, \quad a \in]0,1[.$$

Consequently,

$$D_\mu(a) = \frac{a u_\mu(a)}{a u_\mu(a) + 1} = \nu(]0,a[), \quad a \in]0,1[,$$

and hence, $\mu \in \mathcal{P}_1^0$ and $\Gamma(\mu) = \nu$.
c) We now prove Point 2). Suppose first that $\mu \in \mathcal{P}_1$, $\mu_0 \in \mathcal{P}_1^0$ and $\Gamma(\mu) = \Gamma(\mu_0)$. We then have:

$$u_\mu(a) = u_{\mu_0}(a), \quad a \in]0,1[.$$

By Lemma A.2.4, this entails that there exists $\lambda > 0$ such that:

$$\overline{\mu}(x) = \lambda \overline{\mu}_0(x), \quad x \in]0,1[$$

and therefore, by differentiation, the restriction of μ to $]0,1]$ is equal to $\lambda\mu_0$. Consequently, $\mu = \mu(\{0\})\delta_0 + \lambda\mu_0$ and, since μ is a probability, $\lambda = 1 - \mu(\{0\})$.
Conversely, suppose that $\mu = \mu(\{0\})\delta_0 + (1-\mu(\{0\}))\mu_0$. Since 1 belongs to the support of μ, $\mu(\{0\}) < 1$. Therefore, $\psi_\mu(x) = \psi_{\mu_0}(x)$ for $x \in]0,1]$, and hence, $D_\mu(a) = D_{\mu_0}(a)$ for $a \in]0,1]$, which entails $\Gamma(\mu) = \Gamma(\mu_0)$.

□

Example A.3.4. If ν is a measure which admits a continuous density g which is decreasing on $]0,1[$, and strictly positive in a neighbourhood of 1, then $\nu \in \mathcal{P}'_1$. For example, let us take for $\beta \geq 2\alpha > 0$, $g(x) = \dfrac{\beta - 2\alpha x}{\beta - \alpha} 1_{]0,1[}(x)$. Then, $\nu(]0,x[) = \dfrac{\beta x - \alpha x^2}{\beta - \alpha}$, and some easy computations show that:

$$\overline{\mu}(x) = \frac{\beta - \alpha x}{\beta} (1-x)^{\dfrac{\alpha}{\beta - 2\alpha}} \left(1 - \frac{\alpha x}{\beta - \alpha}\right)^{-\dfrac{\beta - \alpha}{\beta - 2\alpha}}.$$

In particular, letting α tend to 0, we obtain: $\forall x \in [0,1]$, $\overline{\mu}(x) = 1$, i.e. the correspondence:

$$\nu(dx) = 1_{]0,1[}(x)dx \longleftrightarrow \mu(dx) = \delta_1(dx)$$

where δ_1 denotes the Dirac measure at 1.

A.3.3 Examples of elements of $\mathcal{P}'_1$

To a positive measure ρ on $]0,+\infty[$ such that $\int_0^{+\infty} y\rho(dy) < \infty$, we associate the measure:

$$\nu(]0,a[) = \frac{1}{Z} \int_0^{+\infty} (1 - e^{-ay})\rho(dy)$$

where $Z := \displaystyle\int_0^{+\infty} (1 - e^{-y})\rho(dy)$ is such that $\nu(]0,1[) = 1$. Clearly, $a \longmapsto \dfrac{\nu(]0,a[)}{a} = \dfrac{1}{Z} \displaystyle\int_0^{+\infty} e^{-au} \overline{\rho}(u)du$, where $\overline{\rho}(u) = \rho(]u,+\infty[)$, is decreasing and $c_\nu = \dfrac{1}{Z} \displaystyle\int_0^{+\infty} y\rho(dy) < \infty$. Furthermore, $\displaystyle\lim_{a \to 1^-} \frac{\nu([a,1[)}{1-a} = \frac{1}{Z} \int_0^{+\infty} y e^{-y} \rho(dy) > 0$, hence $\displaystyle\int^1 \frac{da}{\nu([a,1[)} = +\infty$, and Theorem A.3.3 applies.

We now give some examples:

i) For $\rho(dx) = e^{-\lambda x}dx$ $(\lambda > 0)$, we obtain: $\nu(]0,a[) = \dfrac{(\lambda+1)a}{\lambda + a}$ $(a \in [0,1])$ and

$$\overline{\mu}(a) = \frac{1}{1-a} \exp\left(-\int_0^a \frac{\lambda+1}{\lambda} \frac{dx}{1-x}\right) = (1-a)^{1/\lambda}$$

ii) For $\rho(dx) = \mathbb{P}(\Gamma \in dx)$ where Γ is a positive r.v. with finite expectation, we obtain

$$\nu(]0,a[) = \mathbb{P}\left(\frac{\mathfrak{e}}{\Gamma} \leq a \Big| \frac{\mathfrak{e}}{\Gamma} \leq 1\right)$$

where $\mathfrak{e}$ is a standard exponential r.v.independent from Γ. In this case, we also note that:

$$\frac{1}{\psi_\mu(a)} = \frac{\nu(]0,a[)}{a} = K\int_0^{+\infty} e^{-ax}\mathbb{P}(\Gamma > x)dx$$
$$= K\mathbb{E}\left[\int_0^{\Gamma} e^{-ax}dx\right] = \frac{K}{a}\mathbb{E}\left[1-e^{-a\Gamma}\right]$$

where $K = 1/\mathbb{E}\left[1-e^{-\Gamma}\right]$. Consequently, the Madan-Yor function

$$D_\mu(a) = \frac{a}{\psi_\mu(a)} = K\mathbb{E}\left[1-e^{-a\Gamma}\right]$$

is the Lévy exponent of a compound Poisson process.

iii) For $\rho(dx) = \dfrac{e^{-\lambda x}}{x}dx$, we obtain: $\nu(]0,a[) = \dfrac{\log(1+a)}{\log(2)}$.

A.3.4 Another presentation of Theorem A.3.1

In the previous Subsection, we have parameterized the measure μ by its tail function $\overline{\mu}(x) := \displaystyle\int_{[x,1]}\mu(dy)$ and its tails ratio u_μ (cf. Lemma A.2.4). Here is another parametrization of μ which provides an equivalent statement to that of Theorem A.3.3.

Theorem A.3.5. *Let μ be a finite positive measure on $[0,1]$ whose support contains 1. Then, μ satisfies $(M{\cdot}Y)$ (i.e. D_μ is increasing on $[0,1]$) if and only if there exists a fonction $\alpha_\mu :]0,1[\longrightarrow \mathbb{R}_+$ such that:*

i) *α_μ is an increasing left-continuous function on $]0,1[$,*

ii) *$\left(\alpha_\mu^2(x)+\alpha_\mu(x)\right)dx - x d\alpha_\mu(x)$ is a positive measure on $]0,1[$,*

iii) $\displaystyle\lim_{x\to 0^+}\frac{\alpha_\mu(x)}{x} < \infty$, *and* $\displaystyle\int^{1^-}\alpha_\mu(x)dx = +\infty$

and such that:

$$\overline{\overline{\mu}}(x) = \overline{\overline{\mu}}(0)\exp\left(-\int_0^x \frac{\alpha_\mu(y)}{y}dy\right). \tag{A.3.4}$$

Properties *i)*, *ii)* and *iii)* are equivalent to the fact that the measure ν, defined on $]0,1[$ by

$$\nu(]0,x[) = \frac{\alpha_\mu(x)}{\alpha_\mu(x)+1}$$

belongs to $\mathcal{P}_1'$. By Theorem A.3.3, this is equivalent to the existence of $\mu\in\mathcal{P}_1$ such that $\Gamma(\mu)=\nu$, which, in turn, is equivalent to

$$u_\mu(x) = \frac{\alpha_\mu(x)}{x},\quad x\in]0,1[,$$

and, finally, is equivalent to (A.3.4).

□

A.4 Some sufficient conditions for $(M\cdot Y)$

Throughout this Section, we consider a positive finite measure μ on $\mathbb{R}_+$ which admits a density, denoted by h. Our aim is to give some sufficient conditions on h which ensure that $(M\cdot Y)$ holds. We start with a general lemma which takes up Madan-Yor condition as given in [MY02, Theorem 4] (this is Condition *iii)* below):

Proposition A.4.1. *Let h be a strictly positive function of $\mathcal{C}^1$ class on $]0,l[$ ($0 < l \leq +\infty$). The three following conditions are equivalent:*

i) For every $c \in]0,1[$, $a \longmapsto \dfrac{h(a)}{h(ac)}$ is a decreasing function.

ii) The function $\varepsilon(y) := -\dfrac{yh'(y)}{h(y)}$ is increasing.

iii) $h(a) = e^{-V(a)}$ where $a \longmapsto aV'(a)$ is an increasing function.

We denote this condition by (S_0).
Moreover, V and ε are related by, for any $a,b \in]0,l[$:

$$V(a) - V(b) = \int_b^a dy \frac{\varepsilon(y)}{y},$$

so that:

$$h(a) = h(b) \exp\left(- \int_b^a \frac{\varepsilon(y)}{y} dy \right).$$

Remark A.4.2. Here are some general observations about condition (S_0):

- if both h_1 and h_2 satisfy condition (S_0), then so does $h_1 h_2$.
- if h satisfies condition (S_0), then, for every $\alpha \in \mathbb{R}$ and $\beta \geq 0$, so does $a \longmapsto a^\alpha h(a^\beta)$.
- As an example, we note that the Laplace transform $h(a) = \mathbb{E}\left[e^{-aX}\right]$ of a positive self-decomposable r.v. X satisfies condition *i)*. Indeed, by definition, for every $c \in [0,1]$, there exists a positive r.v. $X^{(c)}$ independent from X such that:

  $$X \overset{\text{(law)}}{=} cX + X^{(c)}.$$

  Taking Laplace transforms of both sides, we obtain:

  $$h(a) := \mathbb{E}\left[e^{-aX}\right] = \mathbb{E}\left[e^{-acX}\right] \mathbb{E}\left[e^{-aX^{(c)}}\right],$$

  which can be rewritten:

  $$\frac{h(a)}{h(ac)} = \mathbb{E}\left[e^{-aX^{(c)}}\right].$$

- We note that in Theorem 5 of Madan-Yor [MY02], the second and third observations above are used jointly, as the authors remark that the function: $k(a) := \mathbb{E}\left[e^{-a^2X}\right] = h(a^2)$ for X positive and self-decomposable satisfies (S_0).

Proof of Proposition A.4.1

1) $\underline{\text{We prove that } i) \Longleftrightarrow ii)}$

The implication $ii) \Longrightarrow i)$ is clear. Indeed, for $c \in]0,1[$, we write:

$$\frac{h(a)}{h(ac)} = \exp\left(-\int_{ac}^{a} \frac{\varepsilon(y)}{y}dy\right) = \exp\left(-\int_{c}^{1} \frac{\varepsilon(ax)}{x}dx\right) \tag{A.4.1}$$

which is a decreasing function of a since ε is increasing and $0 < c < 1$.

We now prove that $i) \Longrightarrow ii)$. From (A.4.1), we know that for every $c \in]0,1[$, $a \longmapsto \displaystyle\int_{ac}^{a} \frac{\varepsilon(x)}{x}dx$ is an increasing function. Therefore, by differentiation,

$$\forall a \in]0,l[,\ \forall c \in]0,1[, \quad \varepsilon(a) - \varepsilon(ac) \geq 0$$

which proves that ε is an increasing function.

2) $\underline{\text{We prove that } ii) \Longleftrightarrow iii)}$

From the two representations of h, we deduce that $V(a) = \displaystyle\int_{b}^{a} \frac{\varepsilon(y)}{y}dy - \ln h(b)$, which gives, by differentiation:

$$aV'(a) = \varepsilon(a). \tag{A.4.2}$$

This ends the proof of Proposition A.4.1.

□

In the following, we shall once again restrict our attention to probabilities μ on $[0,1]$, and shall assume that they admit a density h which is strictly positive in a neighborhood of 1 (so that 1 belongs to the support of μ). We now give a first set of sufficient conditions (including (S_0)) which encompass most of the examples we shall deal with in the next Section.

Theorem A.4.3. *We assume that the density h of μ is continuous on $]0,1[$. Then, the following conditions imply $(M{\cdot}Y)$:*

(S_0) *h is strictly positive on $]0,1[$ and satisfies condition $i)$ of Proposition A.4.1.*

(S_1) *for every $a \in]0,1[$*

$$\overline{\mu}(a) := \int_{a}^{1} h(x)dx \geq a(1-a)h(a).$$

(S_1') *the function $a \longmapsto a^2h(a)$ is increasing on $]0,1[$.*

(S_2) *the function* $a \longmapsto \log(a\overline{\mu}(a))$ *is concave on* $]0,1[$ *and* $\lim\limits_{a\to 1^-}(1-a)h(a)=0.$

Proof of Theorem A.4.3

1) <u>We first prove: $(S_0) \Longrightarrow (M\cdot Y)$</u>

We write for $a>0$:

$$\frac{1}{D_\mu(a)} = \frac{\int_a^1 yh(y)dy}{a\overline{\mu}(a)} = \frac{[-y\overline{\mu}(y)]_a^1 + \int_a^1 \overline{\mu}(y)dy}{a\overline{\mu}(a)} = 1 + \int_1^{1/a} \frac{\overline{\mu}(ax)}{\overline{\mu}(a)}dx.$$

Clearly, $(M\cdot Y)$ is implied by the property: for all $x>1$, $a \longmapsto \dfrac{\overline{\mu}(ax)}{\overline{\mu}(a)}$ is a decreasing function on $\left]0,\frac{1}{x}\right[$. Differentiating with respect to a, we obtain:

$$\frac{\partial}{\partial a}\left(\frac{\overline{\mu}(ax)}{\overline{\mu}(a)}\right) = \frac{-xh(ax)\overline{\mu}(a) + h(a)\overline{\mu}(ax)}{(\overline{\mu}(a))^2}.$$

We then rewrite the numerator as:

$$\begin{aligned}
&h(a)\int_{ax}^1 h(y)dy - xh(ax)\int_a^1 h(u)du\\
=&xh(a)\int_a^{1/x} h(ux)du - xh(ax)\int_a^1 h(u)du\\
=&xh(a)\int_a^{1/x} h(u)\left(\frac{h(ux)}{h(u)} - \frac{h(ax)}{h(a)}\right)du - xh(ax)\int_{1/x}^1 h(u)du \le 0
\end{aligned}$$

from assertion $i)$ of Proposition A.4.1, since for $x>1$, the function $u \longmapsto \dfrac{h(ux)}{h(u)} = \dfrac{h(ux)}{h\left(ux\frac{1}{x}\right)}$ is decreasing.

2) <u>We now prove: $(S_1) \Longrightarrow (M\cdot Y)$</u>

We must prove that under (S_1), the function $D_\mu(a) := \dfrac{a\overline{\mu}(a)}{\int_a^1 xh(x)dx}$ is increasing. Elementary computations lead, for $a \in]0,1[$, to:

$$\frac{D'_\mu(a)}{D_\mu(a)} = \frac{1}{a} - h(a)\frac{1-D_\mu(a)}{\overline{\mu}(a)}. \tag{A.4.3}$$

From (S_1) and (A.3.1):

$$0 \le \frac{h(a)}{\overline{\mu}(a)}(1-D_\mu(a)) \le \frac{1}{a(1-a)}(1-a) = \frac{1}{a}.$$

Hence, from (A.4.3):

$$\frac{D'_\mu(a)}{D_\mu(a)} \ge \frac{1}{a} - \frac{1}{a} = 0.$$

3) <u>We then prove: $(S_1') \Longrightarrow (S_1)$, hence $(M\cdot Y)$ holds</u>
We have, for $a > 0$:

$$\begin{aligned}
\overline{\mu}(a) := \int_a^1 h(x)dx = \int_a^1 \frac{x^2 h(x)}{x^2}dx \\
&\geq a^2 h(a) \int_a^1 \frac{1}{x^2}dx \quad (\text{since } x \longmapsto x^2 h(x) \text{ is increasing.}) \\
&= a^2 h(a)\left(\frac{1}{a} - 1\right) = ah(a)(1-a).
\end{aligned}$$

4) <u>We finally prove: $(S_2) \Longrightarrow (M\cdot Y)$</u>
We set $\theta(a) = \log(a\overline{\mu}(a))$. Since

$$\int_a^1 th(t)dt = a\overline{\mu}(a) + \int_a^1 \overline{\mu}(t)dt$$

by integration by parts, we have, for $a \in]0,1[$,

$$D_\mu(a) = \frac{e^{\theta(a)}}{e^{\theta(a)} + \int_a^1 \frac{1}{t}e^{\theta(t)}dt}.$$

Therefore, we must prove that the function $a \longmapsto e^{-\theta(a)} \displaystyle\int_a^1 \frac{1}{t}e^{\theta(t)}dt$ is decreasing. Differentiating this function, we need to prove:

$$l(a) := \theta'(a) \int_a^1 \frac{1}{t}e^{\theta(t)}dt + \frac{1}{a}e^{\theta(a)} \geq 0.$$

Now, since $\lim\limits_{a\to 1^-} \theta(a) = -\infty$, an integration by parts gives:

$$l(a) = \int_a^1 \frac{1}{t}e^{\theta(t)}(\theta'(a) - \theta'(t)) + \int_a^1 \frac{1}{t^2}e^{\theta(t)}dt,$$

and, θ' being a decreasing function, this last expression shows that l is also a decreasing function. Therefore, it remains to prove that:

$$\lim_{a\to 1^-} \frac{\overline{\mu}(a) - ah(a)}{\overline{\mu}(a)} \int_a^1 \overline{\mu}(t)dt \geq 0$$

or

$$\lim_{a\to 1^-} \frac{h(a)}{\overline{\mu}(a)} \int_a^1 \overline{\mu}(t)dt = 0.$$

Since $\displaystyle\int_a^1 \overline{\mu}(t)dt \leq (1-a)\overline{\mu}(a)$, the result follows from the assumption $\lim\limits_{a\to 1^-} (1-a)h(a) = 0$.

$\square$

Here are now some alternative conditions which ensure that $(M\cdot Y)$ is satisfied:

Proposition A.4.4. *We assume that μ admits a density h of $\mathcal{C}^1$ class on $]0,1[$ which is strictly positive in a neighbourhood of 1. The following conditions imply $(M{\cdot}Y)$:*

(S_3) *$a \longmapsto a^3h'(a)$ is increasing on $]0,1[$.*

(S_4) *$a \longmapsto a^3h'(a)$ is decreasing on $]0,1[$.*

(S_4') *h is decreasing and concave.*
(Clearly, (S_4') implies (S_4)).

(S_5) *h is a decreasing function and $a \longmapsto \dfrac{ah(a)}{1-a}$ is increasing on $]0,1[$.*

(S_5') *$0 \geq h'(x) \geq -4h(x)$. (In particular, h is decreasing)*

Proof of Proposition A.4.4
1) We first prove: $(S_3) \Longrightarrow (S_1')$
We denote $\ell := \lim\limits_{a\to 0^+} a^3h'(a) \geq -\infty$. If $\ell < 0$, then, there exists $A > 0$ and $\varepsilon \in]0,1[$ such that for $x \in]0,\varepsilon[$, $h'(x) \leq -\dfrac{A}{x^3}$. This implies:

$$h(\varepsilon) - h(x) \leq \frac{A}{2}\left(\frac{1}{\varepsilon^2} - \frac{1}{x^2}\right) \qquad \text{i.e.} \quad h(x) \geq C + \frac{A}{2x^2},$$

which contradicts the fact that $\int_0^1 h(x)dx < \infty$. Therefore $\ell \geq 0$, h is positive and increasing and $h(0^+) := \lim\limits_{x\to 0^+} h(x)$ exists. We then write:

$$\begin{aligned} a^2h(a) = a^2\left(h(0^+) + \int_0^a h'(x)dx\right) &= a^2h(0^+) + a^3\int_0^1 h'(ay)dy \\ &= a^2h(0^+) + \int_0^1 \frac{dy}{y^3}(ay)^3h'(ay), \end{aligned}$$

which implies that $a \longmapsto a^2h(a)$ is increasing as the sum of two increasing functions.

2) We now prove: $(S_4) \Longrightarrow (M{\cdot}Y)$
We set $\widehat{\mu}(a) := \displaystyle\int_{[a,1]} x\mu(dx)$. Thus: $D_\mu(a) := \dfrac{a\overline{\mu}(a)}{\widehat{\mu}(a)}$ and, differentiation shows that $D_\mu'(a) \geq 0$ is equivalent to:

$$\gamma(a) := \overline{\mu}(a)\widehat{\mu}(a) + a^2h(a)\overline{\mu}(a) - ah(a)\widehat{\mu}(a) \geq 0 \qquad a \in]0,1] \qquad \text{(A.4.4)}$$

We shall prove that, under (S_4), $\gamma(1^-) = 0$, $\gamma'(1^-) = 0$ and that γ is convex, which will of course imply that $\gamma \geq 0$ on $]0,1]$. We denote $\ell := \lim\limits_{a\to 1^-} a^3h'(a) \geq -\infty$. Observe first that $h(1^-)$ is finite. Indeed, if

ℓ is finite, then $h'(1^-)$ exists, and so does $h(1^-)$, while if $\ell = -\infty$, then $\lim\limits_{a\to 1^-} h'(a) = -\infty$, hence h is decreasing in the neighborhood of 1 and h being positive, $h(1^-)$ also exists. Therefore, letting $a \to 1$ in (A.4.4), we obtain that $\gamma(1^-) = 0$. Now differentiating (A.4.4), we obtain:

$$\gamma'(a) = -2h(a)\widehat{\mu}(a) + ah(a)\overline{\mu}(a) + ah'(a)(a\overline{\mu}(a) - \widehat{\mu}(a)),$$

and to prove that $\gamma'(1^-) = 0$, we need to show, since $\widehat{\mu}(a) - a\overline{\mu}(a) = \overline{\overline{\mu}}(a)$, that:

$$\lim_{a\to 1^-} h'(a) \int_a^1 \overline{\mu}(t)dt = 0.$$

If $h'(1^-)$ is finite, this property is clearly satisfied. Otherwise $\lim\limits_{a\to 1^-} h'(a) = -\infty$. In this case, we write for a in the neighborhood of 1:

$$0 \leq -h'(a) \int_a^1 \overline{\mu}(t)dt \leq -h'(a)(1-a)\overline{\mu}(a),$$

and it is sufficient to prove that:

$$\lim_{a\to 1^-} (1-a)h'(a) = 0. \tag{A.4.5}$$

Now, since $x \longmapsto x^3 h'(x)$ is decreasing:

$$h(1^-) - h(a) = \int_a^1 h'(x)dx \leq a^3 h'(a) \left[-\frac{1}{2x^2} \right]_a^1 = \frac{a(1+a)}{2} h'(a)(1-a) \leq 0$$

and (A.4.5) follows by passing to the limit as $a \to 1$.
Finally, denote by φ the decreasing continuous function: $a \longmapsto a^3 h'(a)$. Then:

$$\gamma'(a) = -\varphi(a)\frac{\overline{\overline{\mu}}(a)}{a^2} - h(a)\left(\overline{\overline{\mu}}(a) + \widehat{\mu}(a)\right).$$

Consequently, γ' is a continuous function with locally finite variation, and we obtain by differentiation:

$$d\gamma'(a) = -\frac{\overline{\overline{\mu}}(a)}{a^2} d\varphi(a) + h(a)\left(ah(a) + \overline{\mu}(a)\right) da.$$

Hence, $d\gamma'$ is a positive measure on $]0,1[$, which entails that γ is convex on $]0,1[$.

3) <u>We then prove: $(S_5) \Longrightarrow (M{\cdot}Y)$</u>
From (A.4.4), to prove that D_μ is increasing, we need to show that:

$$\rho(a) := \frac{\overline{\mu}(a)\widehat{\mu}(a)}{ah(a)} + a\overline{\mu}(a) - \widehat{\mu}(a) \geq 0.$$

Under (S_5), h is decreasing and hence, for $a \in]0,1[$,

$$\overline{\mu}(a) \leq h(a)(1-a). \tag{A.4.6}$$

Consequently, $\lim_{a\to 1} \rho(a) = 0$, and it is now sufficient to see that $\rho'(a) \leq 0$ on $]0,1[$.

$$\rho'(a) = -\frac{\overline{\mu}(a)\widehat{\mu}(a)}{a^2h^2(a)}\big(h(a) + ah'(a)\big) - \frac{\widehat{\mu}(a)}{a}$$

hence, the assertion $\rho'(a) \leq 0$ on $]0,1[$ is equivalent to:

$$-\frac{1}{ah(a)} - \frac{h'(a)}{h^2(a)} \leq \frac{1}{\overline{\mu}(a)}. \tag{A.4.7}$$

But, under (S_5), $a \longmapsto \dfrac{ah(a)}{1-a}$ is increasing, and therefore we have, for $a \in]0,1[$,

$$\frac{1}{a(1-a)} + \frac{h'(a)}{h(a)} \geq 0. \tag{A.4.8}$$

Then, using (A.4.6) and (A.4.8), we obtain:

$$\begin{aligned}
-\frac{1}{ah(a)} - \frac{h'(a)}{h^2(a)} &\leq -\frac{1}{ah(a)} + \frac{1}{a(1-a)h(a)} \\
&= \frac{1}{ah(a)}\left(\frac{1}{1-a} - 1\right) = \frac{1}{h(a)(1-a)} \leq \frac{1}{\overline{\mu}(a)}
\end{aligned}$$

which gives (A.4.7).

4) <u>We finally prove: $(S'_5) \Longrightarrow (S_5)$</u>

We must prove that $a \longmapsto \dfrac{ah(a)}{1-a}$ is increasing. Differentiating, we obtain:

$$\begin{aligned}
\left(\frac{ah(a)}{1-a}\right)' = \frac{h(a)}{1-a}\left(\frac{1}{a(1-a)} + \frac{h'(a)}{h(a)}\right) &= \frac{h(a)}{1-a}\left(\frac{1}{a(1-a)} - \left|\frac{h'(a)}{h(a)}\right|\right) \\
&\geq \frac{h(a)}{1-a}\left(\frac{1}{a(1-a)} - 4\right) \geq 0
\end{aligned}$$

since, for $a \in [0,1]$, $a(1-a) \leq \frac{1}{4}$.

□

Remark A.4.5. We observe that there exist some implications between these conditions. In particular:

• $(S'_1) \Longrightarrow (S_2)$. Indeed, note first that since (S'_1) implies (S_1), the relation $\overline{\mu}(a) \geq a(1-a)h(a)$ holds, and implies $\lim_{a\to 1^-}(1-a)h(a) = 0$. Then, for

$a \in]0,1[$, condition (S_1') is equivalent to $2h(a) + ah'(a) \geq 0$ and we can write:

$$
\begin{aligned}
-\left(\log\left(a\int_a^1 h(x)dx\right)\right)'' &= \frac{1}{a^2} + \frac{h'(a)\overline{\mu}(a) + h^2(a)}{\overline{\mu}^2(a)} \qquad \text{(A.4.9)}\\
&= \frac{h(a)}{a\overline{\mu}(a)}\left(\frac{\overline{\mu}(a)}{ah(a)} + \frac{ah'(a)}{h(a)} + \frac{ah(a)}{\overline{\mu}(a)}\right)\\
&\geq \frac{h(a)}{a\overline{\mu}(a)}\left(\frac{ah'(a)}{h(a)} + 2\right) \text{ (since for } x \geq 0,\ x + \frac{1}{x} \geq 2)\\
&= \frac{1}{a\overline{\mu}(a)}\big(ah'(a) + 2h(a)\big) \geq 0.
\end{aligned}
$$

This is condition (S_2), i.e. $a \longmapsto \log(a\overline{\mu}(a))$ is a concave function.

- (S_4') implies both (S_0) and (S_5).
  - (S_0) is satisfied since the function $y \longrightarrow \dfrac{yh'(y)}{h(y)}$ is clearly decreasing.
  - To prove that (S_5) is satisfied, we write:

$$
h(1) - h(a) = \int_a^1 h'(x)dx = \int_a^1 \frac{x^2h'(x)}{x^2}dx \leq a^2h'(a)\int_a^1 \frac{dx}{x^2} = ah'(a)(1-a),
$$

hence:

$$
\frac{h(a)}{1-a} + ah'(a) \geq \frac{h(1)}{1-a} \geq 0.
$$

We sum up the implications between these different conditions in the following diagram:

$$
\begin{tikzcd}
 & & & & & (S_4') \arrow[d, Rightarrow] \arrow[dr, Rightarrow] \arrow[dddl, Rightarrow] & (S_5') \arrow[d, Rightarrow] \\
\text{Proposition A.4.4} & & (S_3) \arrow[d, Rightarrow] & & & (S_4) & (S_5) \\
\text{Theorem A.4.3} & & (S_1') \arrow[dl, Rightarrow] \arrow[dr, Rightarrow] & & & & \\
 & (S_1) & & (S_2) & (S_0) & &
\end{tikzcd}
$$

Remark A.4.6.
Let h be a decreasing function with bounded derivative h'. Then, for large enough c, the measure $\mu^{(c)}(dx) := (h(x)+c)dx$ satisfies condition (S_5'), hence $(M{\cdot}Y)$. Indeed, for $h^{(c)}(x) = h(x) + c$, we have:

$$
\left|\frac{h^{(c)\prime}(x)}{h^{(c)}(x)}\right| = \frac{|h'(x)|}{h(x)+c} \xrightarrow[c\to+\infty]{} 0
$$

This convergence being uniform, for large enough c, we obtain:

$$\sup_{x\in[0,1]} \left|\frac{h^{(c)\prime}(x)}{h^{(c)}(x)}\right| \leq 4.$$

A.5 Case where the support of μ is $\mathbb{R}_+$

In this Section, we assume that $\mu(dx) = h(x)dx$ is a positive measure whose density h is strictly positive a.e. on $\mathbb{R}_+$. The following theorem gives sufficient conditions on h for the function D_μ to be increasing and converging to 1 when a tends to $+\infty$.

Theorem A.5.1.
We assume that μ admits a density h on $\mathbb{R}_+$ which satisfies (S_0) (see Proposition A.4.1).
1) *Then, there exists $\rho > 2$ (possibly $+\infty$) such that:*

$$\forall\ c \in]0,1[, \qquad \lim_{a\to+\infty} \frac{h(a)}{h(ac)} = c^\rho. \tag{A.5.1}$$

Furthermore:

$$\rho = \lim_{a\to+\infty} \varepsilon(a) = \lim_{a\to+\infty} aV'(a).$$

2) *D_μ is an increasing function which converges towards ℓ with:*

- *if $\rho < +\infty$, then $\ell = \dfrac{\rho-2}{\rho-1}$*
- *if $\rho = +\infty$, then $\ell = 1$.*

In particular, if $\rho = +\infty$, then, there exists a probability measure ν_μ such that:

$$D_\mu(a) = \nu_\mu(]0,a[), \qquad a \geq 0.$$

Remark A.5.2.
- Point 1) of Theorem A.5.1 casts a new light on Proposition A.4.1. Indeed, from (A.5.1), we see that h is a regularly varying function in the sense of Karamata, and Proposition A.4.1 looks like a version of Karamata's representation Theorem (see [BGT89, Chapter 1, Theorems 1.3.1 and 1.4.1]).
- The property that the function $a \longmapsto \dfrac{h(a)}{h(ac)}$ is decreasing is not necessary to obtain the limit of D_μ, see [BGT89, Theorem 8.1.4].

Proof of Theorem A.5.1
1) We first prove Point 1)

We assume that h satisfies (S_0) on $\mathbb{R}_+$. Therefore the decreasing limit $\gamma_c := \lim_{a\to+\infty} \dfrac{h(a)}{h(ac)}$ exists and belongs to $[0,1]$. Then, for all $c,d \in]0,1[$:

$$\gamma_{cd} = \lim_{a\to+\infty} \frac{h(a)}{h(acd)} = \lim_{a\to+\infty} \frac{h(a)}{h(ac)} \frac{h(ac)}{h(acd)} = \gamma_c\gamma_d.$$

This implies that $\gamma_c = c^\rho$ with $\rho \in \mathbb{R}_+$. Now, let $\eta(a) = \int_a^{+\infty} yh(y)dy$. For $A > 1$, we have

$$\begin{aligned}
\eta(a) = \int_a^{+\infty} yh(y)dy = a^2 \int_1^{+\infty} zh(az)dz &\geq a^2 \int_1^A z\frac{h(az)}{h(z)}h(z)dz \\
&\geq a^2 \frac{h(aA)}{h(A)} \int_1^A zh(z)dz \\
&\xrightarrow[A\to+\infty]{} a^{2-\rho} \int_1^{+\infty} zh(z)dz.
\end{aligned}$$

Letting a tend to $+\infty$, we obtain, since $\eta(a) \xrightarrow[a\to+\infty]{} 0$, that necessarily $\rho > 2$. Then, passing to the limit in (A.4.1), we obtain:

$$c^\rho = \exp\left(-\int_c^1 \frac{\varepsilon(+\infty)}{y}dy\right), \quad \text{i.e. } \varepsilon(+\infty) = \rho.$$

The last equality is a direct consequence of (A.4.2).

2) <u>We now prove that D_μ is increasing and converges towards ℓ</u>

As in Theorem A.2.3, we denote $\overline{\mu}(a) = \int_a^{+\infty} h(y)dy$. Then:

$$\frac{1}{D_\mu(a)} = \frac{[-y\overline{\mu}(y)]_a^{+\infty} + \int_a^{+\infty} \overline{\mu}(y)dy}{a\overline{\mu}(a)} = 1 + \int_1^{+\infty} \frac{\overline{\mu}(ax)}{\overline{\mu}(a)}dx. \tag{A.5.2}$$

Now, the proof of the increase of D_μ is exactly the same as that of the implication $S_0 \Longrightarrow (M\cdot Y)$ (see Theorem A.4.3). Then, D_μ being bounded by 1, it converges towards a limit ℓ, and it remains to identify ℓ. We write, for $x > 1$:

$$\frac{\overline{\mu}(a)}{\overline{\mu}(ax)} = \frac{\displaystyle\int_a^{+\infty} h(y)dy}{\displaystyle\int_{ax}^{+\infty} h(y)dy} = \frac{\displaystyle\int_1^{+\infty} h(au)du}{\displaystyle\int_x^{+\infty} h(au)du}$$

$$= \frac{\displaystyle\int_1^{x} h(au)du}{\displaystyle\int_x^{+\infty} h(au)du} + 1$$

$$= \frac{\displaystyle\int_1^{x} \frac{h(ax\frac{u}{x})}{h(ax)}du}{\displaystyle\int_x^{+\infty} \frac{h(au)}{h\left(au\frac{x}{u}\right)}du} + 1 \xrightarrow[a\to+\infty]{} \frac{\displaystyle\int_1^{x} \left(\frac{x}{u}\right)^{\rho} du}{\displaystyle\int_x^{+\infty} \left(\frac{x}{u}\right)^{\rho} du} + 1$$

from (A.5.1). Now, we must discuss different cases:

$-$ if $\rho = +\infty$, then $\lim\limits_{a\to+\infty} \dfrac{\overline{\mu}(a)}{\overline{\mu}(ax)} = +\infty$, and plugging this limit into (A.5.2), we obtain $\ell = 1$.

$-$ if $\rho < +\infty$, we obtain:

$$\lim_{a\to+\infty} \frac{\overline{\mu}(a)}{\overline{\mu}(ax)} = \frac{1}{x^{1-\rho}}.$$

Plugging this into (A.5.2), we obtain:

$$\frac{1}{\ell} = 1 + \int_1^{+\infty} \frac{dx}{x^{\rho-1}} = 1 - \frac{1}{2-\rho} = \frac{1-\rho}{2-\rho}.$$

$\square$

Remark A.5.3.
More generally, for $p \geq 1$, there is the equivalence:

$$\int^{+\infty} y^p h(y)dy < \infty \Longleftrightarrow \rho > p+1.$$

The implication $\Longrightarrow$ can be proven in exactly the same way as Point 1). Conversely, since $\varepsilon(y)$ tends to ρ when y tends to $+\infty$, there exists $A > 0$ and $\theta > 0$ such that: $\forall y \geq A$, $\varepsilon(y) \geq p+1+\theta$. Then applying Proposition A.4.1, we obtain:

$$h(a) = h(A)\exp\left(-\int_A^a \frac{\varepsilon(y)}{y}dy\right) \leq h(A)\exp\left(-(p+1+\theta)\int_A^a \frac{dy}{y}\right)$$

$$= h(A)\left(\frac{A}{a}\right)^{p+1+\theta}.$$

We note in particular that μ admits moments of all orders if and only if $\rho = +\infty$.

A.6 Examples

We take $\mu(dx) = h(x)dx$ and give some examples of functions h which enjoy the $(M\cdot Y)$ property. For some of them, we draw the graphs of h, D_μ, u_μ and $a \longmapsto \dfrac{\nu_\mu(]0,a[)}{a}$.

A.6.1 Beta densities $h(x) = x^\alpha(1-x)^\beta 1_{]0,1[}(x)$ $(\alpha, \beta > -1)$

$i)$ For $-1 < \beta \le 0$ (and $\alpha > -1$), the function $x \longmapsto x^2 h(x)$ is increasing, hence from (S_1'), condition $(M\cdot Y)$ holds.
$ii)$ For $\beta \ge 0$:

$$\frac{h(a)}{h(ac)} = \frac{1}{c^\alpha}\left(1 - \frac{a(1-c)}{1-ac}\right)^\beta$$

which, for $0 < c < 1$, is a decreasing function of a, hence condition (S_0) is satisfied and $(M\cdot Y)$ also holds in that case.

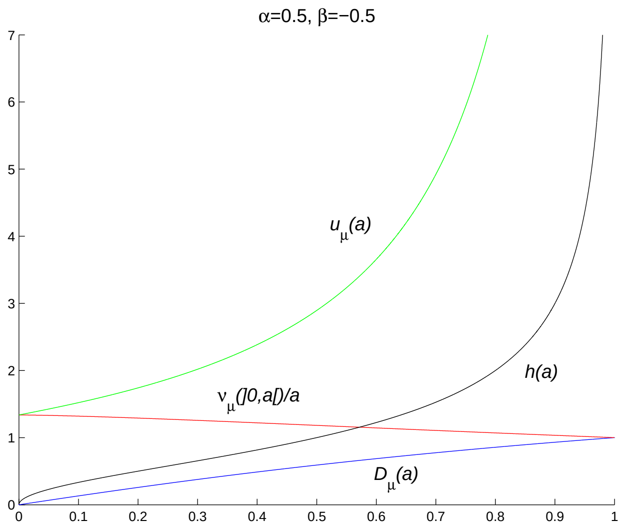

Figure 1: Graphs for $h(x) = \sqrt{\dfrac{x}{(1-x)}}1_{[0,1[}(x)$

A.6.2 Further examples

– The function $h(x) = \dfrac{x^\alpha}{(1+x^2)^\beta}1_{[0,1]}(x)$ $(\alpha > -1, \beta \in \mathbb{R})$ satisfies $(M\cdot Y)$. Indeed, for $\beta \le 0$, $x \longmapsto x^2 h(x)$ is an increasing function on $[0,1]$, hence condition (S_1') holds, while, for $\beta \ge 0$, condition (S_0) is satisfied.
– The function $h(x) = \dfrac{x^\alpha}{(1-x^2)^\beta}1_{[0,1]}(x)$ $(\alpha > -1, \beta < 1)$ satisfies $(M\cdot Y)$.

As in the previous example, for $0 \leq \beta \leq 1$, the function $x \longmapsto x^2 h(x)$ is increasing on $[0,1]$, and for $\beta \leq 0$, this results from condition (S_0).

A.6.3 $\quad h(x) = |\cos(\pi x)|^m 1_{[0,1]}(x)\ (m \in \mathbb{R}_+)$

We check that this example satisfies condition (S_1). Indeed, for $a \geq \frac{1}{2}$, $a \longmapsto h(a)$ is increasing, hence:

$$\int_a^1 |\cos(\pi x)|^m dx \geq |\cos(\pi a)|^m (1-a) \geq a|\cos(\pi a)|^m (1-a).$$

For $a \leq \frac{1}{2}$ we write by symmetry:

$$\begin{aligned} \int_a^1 |\cos(\pi x)|^m dx &= \int_0^{1-a} |\cos(\pi x)|^m dx \\ &\geq \int_0^a |\cos(\pi x)|^m dx \\ &\geq a|\cos(\pi a)|^m \geq a|\cos(\pi a)|^m (1-a). \end{aligned}$$

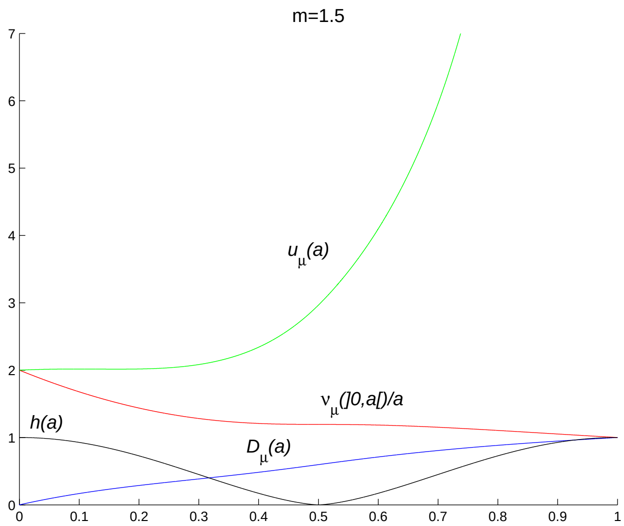

Figure 2: Graphs for $h(x) = |\cos(\pi x)|^{3/2} 1_{[0,1]}(x)$

Remark A.6.1. More generally, every function which is symmetric with respect to the axis $x = \frac{1}{2}$, and is first decreasing and then increasing, satisfies condition (S_1).

A.6.4 $\quad h(x) = x^\alpha e^{-x^\lambda} 1_{]0,1]}(x)\ (\alpha > -1, \lambda \in \mathbb{R})$

This is a direct consequence of (S_0).

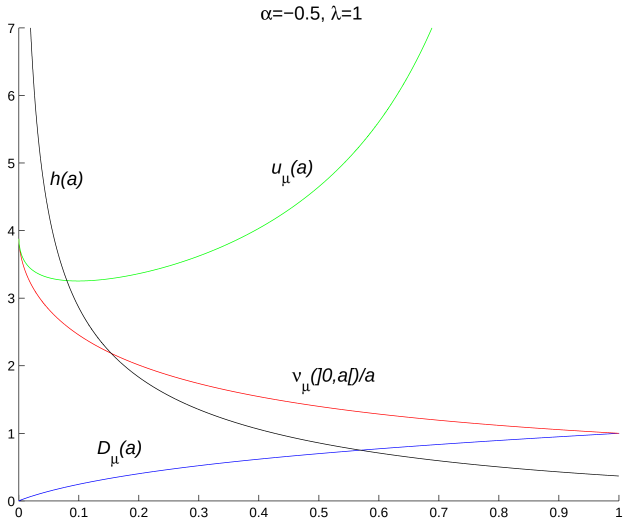

Figure 3: Graphs for $h(x) = \dfrac{e^{-x}}{\sqrt{x}} 1_{]0,1]}(x)$

A.6.5 An example where $(M{\cdot}Y)$ is not satisfied

Let μ be the measure with density h defined by:

$$h(x) = c1_{[0,p[}(x) + e1_{[p,1]}(x) \qquad (c, e \geq 0, p \in]0,1[).$$

For $a < p$, it holds:

$$D_\mu(a) := \frac{2a\left(c(p-a) + e(1-p)\right)}{c(p^2 - a^2) + e(1-p^2)}$$

D_μ is $\mathcal{C}^\infty$ on $[0,p[$, and, for $a < p$, we have:

$$D'_\mu(a) = 2\frac{c^2p(p-a)^2 + e^2(1-p)^2(1+p) + ec(1-p)\left((p-a)^2 + p^2 + p - 2a\right)}{\left(c(p^2-a^2) + e(1-p^2)\right)^2}$$

and

$$D'_\mu(p^-) = 2\frac{e^2(1-p)^2\left(1 + p\left(1 - \frac{c}{e}\right)\right)}{e^2(1-p^2)^2} = 2\frac{1 + p\left(1 - \frac{c}{e}\right)}{(1+p)^2}.$$

Therefore, it is clear that, for $\dfrac{c}{e}$ large enough, $D'_\mu(p^-) < 0$, hence D_μ is not increasing on $[0,1]$. Note that, if $e \geq c$ (h is increasing), then $D'_\mu \geq 0$ (see condition (S'_1)), and that D_μ is increasing if and only if $D'_\mu(p^-) \geq 0$, i.e. $\dfrac{c}{e} \leq 1 + \dfrac{1}{p}$.

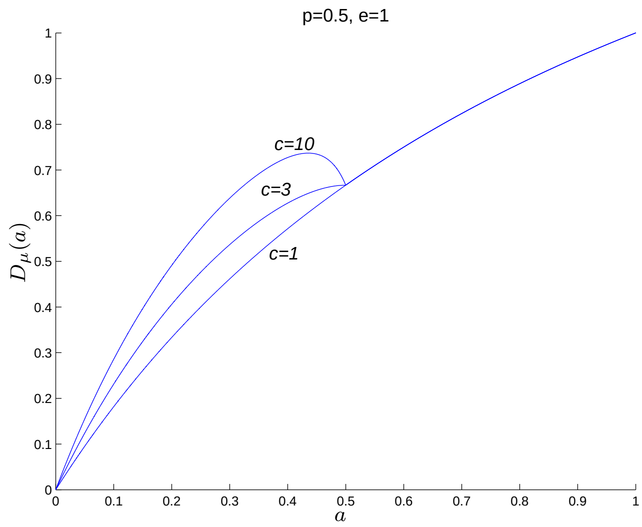

Figure 4: Graph of D_μ for $h(x) = c1_{[0,1/2[}(x) + 1_{[1/2,1]}(x)$

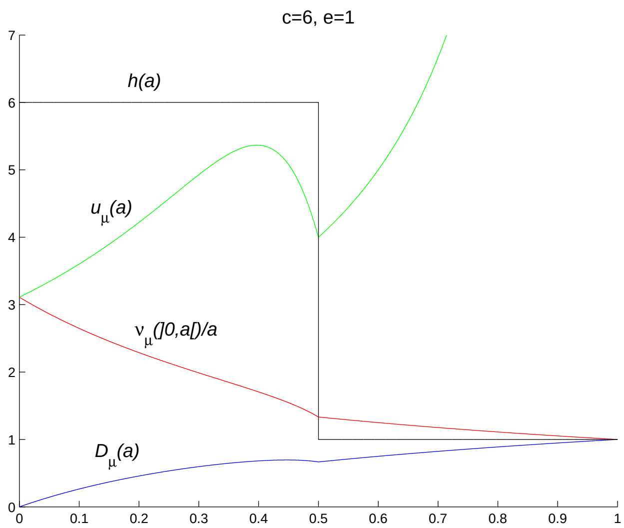

Figure 5: Graphs for $h(x) = 61_{[0,1/2[}(x) + 1_{[1/2,1]}(x)$

A.6.6 A situation where neither condition $(S_i)_{i=0\ldots5}$ is satisfied, but $(M{\cdot}Y)$ is

Let h be a function such that, for $a \in [1/2, 1]$, $D'_\mu(a) > 0$. We define h on $[0, 1/2]$ such that $\int_0^{1/2} h(x)dx < \varepsilon$ and $\sup\limits_{x\in[0,1/2]} h(x) \leq \eta$. Then, for $\varepsilon > 0$ and $\eta \geq 0$ small enough, the measure $\mu(dx) = h(x)dx$ satisfies $(M{\cdot}Y)$ and h may be chosen in such a way that none of the preceding conditions is satisfied.

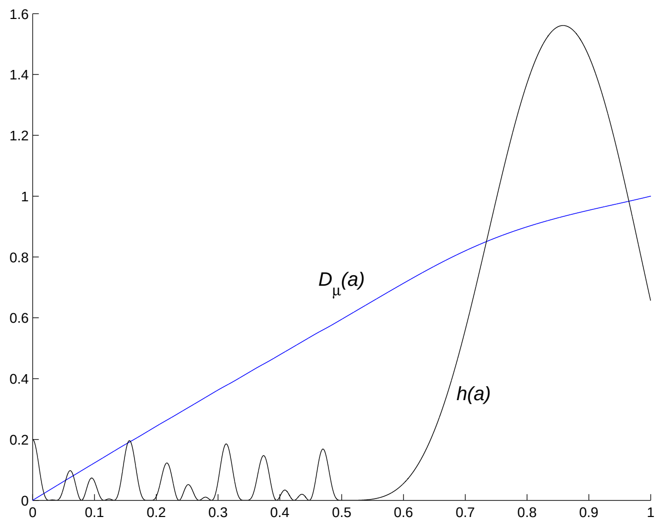

Figure 6: Graphs for h satisfying neither condition $(S_i)_{i=0\ldots5}$

Part B

Construction of randomized Skorokhod embeddings

B.1 Introduction

In this Part B, our aim is still to construct martingales satisfying the properties **(a)** and **(b)**, this time by a (seemingly) new Skorokhod embedding method, in the spirit of the original Skorokhod method, and of the so-called Hall method (see [Obl04] for comments and references; we also thank J. Obłój [Obl09] for writing a short informal draft about this method).
Our method of randomized Skorokhod embedding will ensure directly that the family of stopping times $(\tau_t\ ,\ t \geq 0)$ is increasing.
Here is the content of this Part B:

- In Section B.2, we consider a real valued, integrable and centered random variable X. We prove that there exist an $\mathbb{R}_+$-valued random variable V and an $\mathbb{R}_-^*$-valued random variable W, with V and W independent and independent of $(B_u\ ,\ u \geq 0)$, such that, denoting:

  $$\tau = \inf\{u \geq 0\ ;\ B_u = V \text{ or } B_u = W\}\ ,$$

  Property **(Sk1)** is satisfied by this randomized stopping time τ, i.e: $B_\tau \overset{\text{(law)}}{=} X$. To prove this result we use, as an essential tool, the Schauder-Tychonoff fixed point theorem.

- In Section B.3, we prove that the stopping time τ defined in Section B.2 satisfies **(Sk2)**, i.e: the martingale $B^\tau := (B_{t\wedge\tau}, u \geq 0)$ is uniformly integrable. Moreover, for every $p \geq 1$, we state conditions ensuring that B^τ is a martingale belonging to the space H^p consisting of all martingales $(M_t, t \geq 0)$ such that $\sup_{t\geq 0} |M_t| \in L^p$.

- In Section B.4, we follow the method presented in the general introduction, and construct an increasing family of randomized stopping times $(\tau_t\ ,\ t \geq 0)$, such that $(B_{\tau_t}\ ,\ t \geq 0)$ is a martingale satisfying properties **(a)** and **(b)**.

B.2 Randomized Skorokhod embedding

B.2.1 Notation

We denote by $\mathbb{R}_+$ (resp. $\mathbb{R}_-^*$) the interval $[0,+\infty[$ (resp. $]-\infty,0[$), and by $\mathcal{M}_+$ (resp. $\mathcal{M}_-$) the set of positive finite measures on $\mathbb{R}_+$ (resp. $\mathbb{R}_-^*$), equipped with the weak topology:

$$\sigma(\mathcal{M}_+ \, , \, \mathcal{C}^0(\mathbb{R}_+)) \qquad (\text{resp. } \sigma(\mathcal{M}_- \, , \, \mathcal{C}^0(\mathbb{R}_-^*)) \,)$$

where $\mathcal{C}^0(\mathbb{R}_+)$ (resp. $\mathcal{C}^0(\mathbb{R}_-^*)$) denotes the space of continuous functions on $\mathbb{R}_+$ (resp. $\mathbb{R}_-^*$) tending to 0 at $+\infty$ (resp. at 0 and at $-\infty$).
$B=(B_u \, , \; u \geq 0)$ denotes a standard Brownian motion started from 0.
In the sequel we consider a real valued, integrable, centered random variable X, the law of which we denote by μ . The restrictions of μ to $\mathbb{R}_+$ and $\mathbb{R}_-^*$ are denoted respectively by μ_+ and μ_- .

B.2.2 Existence of a randomized stopping time

This subsection is devoted to the proof of the following Skorokhod embedding method.

Theorem B.2.1.

i) There exist an $\mathbb{R}_+$-valued random variable V and an $\mathbb{R}_-^*$-valued random variable W, V and W being independent and independent of $(B_u \, , \; u \geq 0)$, such that, setting

$$\tau = \inf\{u \geq 0 \; ; \; B_u = V \;\; or \;\; B_u = W\} \, ,$$

one has: $B_\tau \overset{(law)}{=} X$.

ii) Denoting by γ_+ (resp. γ_-) the law of V (resp. W), then:

$$\mu_+ \leq \gamma_+ \ll \mu_+ \qquad and \qquad \mu_- \leq \gamma_- \ll \mu_- \, .$$

Moreover,

$$\mathbb{E}[V \wedge (-W)] \leq \mathbb{E}[|X|] \leq 2\,\mathbb{E}[V \wedge (-W)] \tag{B.2.1}$$

and, for every $p > 1$,

$$\frac{1}{2}\mathbb{E}\left[(V \wedge (-W))\left(V^{p-1} + (-W)^{p-1}\right)\right] \\ \leq \mathbb{E}[|X|^p] \leq \mathbb{E}\left[(V \wedge (-W))\left(V^{p-1} + (-W)^{p-1}\right)\right] \tag{B.2.2}$$

Proof of Theorem B.2.1
In the following, we exclude the case $\mu = \delta_0$, the Dirac measure at 0. Otherwise, it suffices to set: $V = 0$. Then, *i)* is satisfied since $\tau = 0$, and *ii)* is also satisfied except the property $\gamma_- \ll \mu_-$ (since $\mu_- = 0$).

1. We first recall the following classical result: Let $b < 0 \le a$ and

$$T_{b,a} = \inf\{u \ge 0 \; ; \; B_u = a \;\text{ or }\; B_u = b\} \; .$$

Then,

$$P(B_{T_{b,a}} = a) = \frac{-b}{a-b} \qquad \text{and} \qquad P(B_{T_{b,a}} = b) = \frac{a}{a-b} \; .$$

2. Let V and W be respectively an $\mathbb{R}_+$-valued random variable and an $\mathbb{R}_-^*$-valued random variable, V and W being independent and independent of B, and let τ, γ_+, γ_- be defined as in the statement of the theorem. As a direct consequence of Point 1, we obtain that $B_\tau \overset{(\text{law})}{=} X$ if and only if:

$$\mu_+(\mathrm{d}v) = \left(\int_{\mathbb{R}_-^*} \frac{-w}{v-w} \, \gamma_-(\mathrm{d}w) \right) \gamma_+(\mathrm{d}v) \quad \text{on } \mathbb{R}_+ \tag{B.2.3}$$

$$\mu_-(\mathrm{d}w) = \left(\int_{\mathbb{R}_+} \frac{v}{v-w} \, \gamma_+(\mathrm{d}v) \right) \gamma_-(\mathrm{d}w) \quad \text{on } \mathbb{R}_-^* \tag{B.2.4}$$

As γ_+ and γ_- are probabilities, the above equations entail:

$$\gamma_+(\mathrm{d}v) = \mu_+(\mathrm{d}v) + \left(\int_{\mathbb{R}_-^*} \frac{v}{v-w} \, \gamma_-(\mathrm{d}w) \right) \gamma_+(\mathrm{d}v) \quad \text{on } \mathbb{R}_+ \tag{B.2.5}$$

$$\gamma_-(\mathrm{d}w) = \mu_-(\mathrm{d}w) + \left(\int_{\mathbb{R}_+} \frac{-w}{v-w} \, \gamma_+(\mathrm{d}v) \right) \gamma_-(\mathrm{d}w) \quad \text{on } \mathbb{R}_-^* \tag{B.2.6}$$

To prove Point $i)$ of the theorem, we shall now solve this system of equations (B.2.5) and (B.2.6) by a fixed point method, and then we shall verify that the solution thus obtained is a pair of probabilities, which will entail (B.2.3) and (B.2.4).

3. We now introduce some further notation. If $(a,b) \in \mathcal{M}_+ \times \mathcal{M}_-$ and $\varepsilon > 0$, we set

$$a(\varepsilon) = \int 1_{]0,\varepsilon[}(v) \, a(\mathrm{d}v) \qquad \text{and} \qquad b(\varepsilon) = \int 1_{]-\varepsilon,0[}(w) \, b(\mathrm{d}w) \; .$$

We also set: $m_+ = \int \mu_+(\mathrm{d}v)$, $m_- = \int \mu_-(\mathrm{d}w)$. We note that, since μ is centered and is not the Dirac measure at 0, then $m_+ > 0$ and $m_- > 0$. We then define:

$$\rho(\varepsilon) := 4 \sup \left(\mu_+(\varepsilon) \, m_+^{-1}, \mu_-(\varepsilon) \, m_-^{-1} \right)$$

and

$$\begin{aligned} \Theta \;\; := \;\; & \{(a,b) \in \mathcal{M}_+ \times \mathcal{M}_- \; ; \; a \ge \mu_+, b \ge \mu_-, \int a(\mathrm{d}v) + \int b(\mathrm{d}w) \le 2 \\ & \text{and for every } \varepsilon \le \varepsilon_0, \;\; a(\varepsilon) \le \rho(\varepsilon) \;\text{ and }\; b(\varepsilon) \le \rho(\varepsilon)\} \end{aligned}$$

where ε_0 will be defined subsequently.
Finally, we define $\Gamma = (\Gamma_+, \Gamma_-) : \mathcal{M}_+ \times \mathcal{M}_- \longrightarrow \mathcal{M}_+ \times \mathcal{M}_-$ by:

$$\begin{aligned}
\Gamma_+(a,b)(\mathrm{d}v) &= \mu_+(\mathrm{d}v) + \left(\int_{\mathbb{R}_-^*} \frac{v}{v-w}\, b(\mathrm{d}w) \right) a(\mathrm{d}v) \\
\Gamma_-(a,b)(\mathrm{d}w) &= \mu_-(\mathrm{d}w) + \left(\int_{\mathbb{R}_+} \frac{-w}{v-w}\, a(\mathrm{d}v) \right) b(\mathrm{d}w)
\end{aligned}$$

Lemma B.2.1. Θ *is a convex compact subset of* $\mathcal{M}_+ \times \mathcal{M}_-$ *(equipped with the product of the weak topologies), and* $\Gamma(\Theta) \subset \Theta$ *.*

Proof of Lemma B.2.1
The first part is clear. Suppose that $(a,b) \in \Theta$. By definition of Γ, we have:

$$\Gamma_+(a,b) \geq \mu_+ \ , \ \Gamma_-(a,b) \geq \mu_-$$

and

$$\int \Gamma_+(a,b)(\mathrm{d}v) + \int \Gamma_-(a,b)(\mathrm{d}w) = 1 + \left(\int a(\mathrm{d}v) \right) \left(\int b(\mathrm{d}w) \right) \tag{B.2.7}$$

Consequently,

$$\int \Gamma_+(a,b)(\mathrm{d}v) + \int \Gamma_-(a,b)(\mathrm{d}w) \leq 2$$

and

$$\int \Gamma_+(a,b)(\mathrm{d}v) \leq 2 - m_- \ , \ \int \Gamma_-(a,b)(\mathrm{d}w) \leq 2 - m_+ \tag{B.2.8}$$

On the other hand,

$$\begin{aligned}
\Gamma_+(a,b)(\varepsilon) &= \mu_+(\varepsilon) + \int 1_{]0,\varepsilon[}(v)\, a(\mathrm{d}v) \int 1_{]-v,0]}(w)\, \frac{v}{v-w}\, b(\mathrm{d}w) \\
&\quad + \int 1_{]0,\varepsilon[}(v)\, a(\mathrm{d}v) \int 1_{]-\infty,-v]}(w)\, \frac{v}{v-w}\, b(\mathrm{d}w) \ .
\end{aligned}$$

Since $\dfrac{v}{v-w} \leq 1$, and $\dfrac{v}{v-w} \leq 1/2$ if $w \leq -v$, taking into account (B.2.8) we obtain:

$$\Gamma_+(a,b)(\varepsilon) \leq \mu_+(\varepsilon) + a(\varepsilon)\, b(\varepsilon) + a(\varepsilon) \left(1 - \frac{m_+}{2} \right) \ .$$

Hence,

$$\Gamma_+(a,b)(\varepsilon) \leq \rho^2(\varepsilon) + \rho(\varepsilon) \left(1 - \frac{m_+}{2} \right) + \mu_+(\varepsilon) \ .$$

In order to deduce from the preceding that: $\Gamma_+(a,b)(\varepsilon) \le \rho(\varepsilon)$, it suffices to prove:

$$\rho^2(\varepsilon) - \frac{m_+}{2}\,\rho(\varepsilon) + \mu_+(\varepsilon) \le 0$$

or

$$\rho(\varepsilon) \in \left[\frac{1}{4}(m_+ - \sqrt{m_+^2 - 16\,\mu_+(\varepsilon)}), \frac{1}{4}(m_+ + \sqrt{m_+^2 - 16\,\mu_+(\varepsilon)})\right] ,$$

which is satisfied for $\varepsilon \le \varepsilon_0$ for some choice of ε_0, by definition of ρ. The proof of $\Gamma_-(a,b)(\varepsilon) \le \rho(\varepsilon)$ is similar.

$\square$

Lemma B.2.2. *The restriction of the map Γ to Θ is continuous.*

Proof of Lemma B.2.2
We first prove the continuity of Γ_+. For $\varepsilon > 0$, we denote by h_ε a continuous function on $\mathbb{R}_-^*$ satisfying:

$$h_\varepsilon(w) = 0 \;\text{ for }\; -\varepsilon < w < 0 \;\;,\;\; h_\varepsilon(w) = 1 \text{ for } w < -2\,\varepsilon$$

and, for every $w < 0$, $0 \le h_\varepsilon(w) \le 1$. We set: $\Gamma_+^\varepsilon(a,b) = \Gamma_+(a, h_\varepsilon\, b)$. Then, $\Gamma_+^\varepsilon(a,b) \le \Gamma_+(a,b)$ and

$$0 \le \int \Gamma_+(a,b)(\mathrm{d}v) - \int \Gamma_+^\varepsilon(a,b)(\mathrm{d}v) \le 2\,\rho(2\,\varepsilon) \;,$$

which tends to 0 as ε tends to 0. Therefore, by uniform approximation, it suffices to prove the continuity of the map Γ_+^ε.

Let (a_n, b_n) be a sequence in Θ, weakly converging to (a,b), and let $\varphi \in \mathcal{C}^0(\mathbb{R}_+)$. It is easy to see that the set:

$$\left\{\frac{v\,\varphi(v)}{v - \bullet}\, h_\varepsilon(\bullet) \;;\; v \ge 0\right\}$$

is relatively compact in the Banach space $\mathcal{C}^0(\mathbb{R}_-^*)$. Consequently,

$$\lim_{n\to\infty} \int \frac{v\,\varphi(v)}{v-w}\, h_\varepsilon(w)\; b_n(\mathrm{d}w) = \int \frac{v\,\varphi(v)}{v-w}\, h_\varepsilon(w)\; b(\mathrm{d}w) \tag{B.2.9}$$

uniformly with respect to v. Since

$$\left|\int \frac{v\,\varphi(v)}{v-w}\, h_\varepsilon(w)\; b_n(\mathrm{d}w)\right| \le 2\,|\varphi(v)| \;, \tag{B.2.10}$$

then

$$\left\{\int \frac{v\,\varphi(v)}{v-w}\, h_\varepsilon(w)\; b_n(\mathrm{d}w) \;;\; n \ge 0\right\}$$

is relatively compact in the Banach space $\mathcal{C}^0(\mathbb{R}_+)$. Therefore,

$$\lim_{n\to\infty} \int \varphi(v)\, \Gamma_+^\varepsilon(a_n, b_p)(\mathrm{d}v) = \int \varphi(v)\, \Gamma_+^\varepsilon(a, b_p)(\mathrm{d}v)$$

uniformly with respect to p, and, by (B.2.9) and (B.2.10):

$$\lim_{n\to\infty} \int \varphi(v)\, \Gamma_+^\varepsilon(a, b_n)(\mathrm{d}v) = \int \varphi(v)\, \Gamma_+^\varepsilon(a, b)(\mathrm{d}v) \ .$$

Finally,

$$\lim_{n\to\infty} \int \varphi(v)\, \Gamma_+^\varepsilon(a_n, b_n)(\mathrm{d}v) = \int \varphi(v)\, \Gamma_+^\varepsilon(a, b)(\mathrm{d}v) \ ,$$

which proves the desired result.

The proof of the continuity of Γ_- is similar, but simpler since it does not need an approximation procedure.

□

As a consequence of Lemma B.2.1 and Lemma B.2.2, we may apply the Schauder-Tychonoff fixed point theorem (see, for instance, [DS88, Theorem V.10.5]), which yields the existence of a pair $(\gamma_+, \gamma_-) \in \Theta$ satisfying (B.2.5) and (B.2.6). We set

$$\alpha_+ = \int \gamma_+(\mathrm{d}v) \ , \ \alpha_- = \int \gamma_-(\mathrm{d}w)$$

and we shall now prove that $\alpha_+ = \alpha_- = 1$.

4. By (B.2.7) applied to $(a, b) = (\gamma_+, \gamma_-)$, we obtain:

$$\alpha_+ + \alpha_- = 1 + \alpha_+\, \alpha_-$$

and therefore, $\alpha_+ = 1$ or $\alpha_- = 1$. Suppose, for instance, $\alpha_+ = 1$. Since $\alpha_+ + \alpha_- \leq 2$, then $\alpha_- \leq 1$. We now suppose $\alpha_- < 1$. By (B.2.5), $\gamma_+ \leq \mu_+ + \alpha_-\, \gamma_+$, and hence, $\gamma_+ \leq (1 - \alpha_-)^{-1}\, \mu_+$. Consequently,

$$\int v\, \gamma_+(\mathrm{d}v) \leq (1 - \alpha_-)^{-1} \int v\, \mu_+(\mathrm{d}v) < \infty \ .$$

We deduce from (B.2.5) and (B.2.6) that, for every $r > 0$,

$$\int_0^\infty v\, \gamma_+(\mathrm{d}v) + \int_{-r}^0 w\, \gamma_-(\mathrm{d}w) \\ = \varepsilon_1(r) + \varepsilon_2(r) + \int_0^\infty \gamma_+(\mathrm{d}v) \int_{-r}^0 \gamma_-(\mathrm{d}w)(v + w) \quad \text{(B.2.11)}$$

with

$$\varepsilon_1(r) = \int_{-r}^{+\infty} x\,\mu(\mathrm{d}x) \qquad \text{and}$$

$$\varepsilon_2(r) = \int_0^{\infty} \gamma_+(\mathrm{d}v) \int_{-\infty}^{-r} \gamma_-(\mathrm{d}w) \frac{v^2}{v-w} \ .$$

Since X is centered, $\lim_{r\to+\infty} \varepsilon_1(r) = 0$. On the other hand,

$$\varepsilon_2(r) \le \left(\int v\,\gamma_+(\mathrm{d}v) \right) \left(\int_{-\infty}^{-r} \gamma_-(\mathrm{d}w) \right)$$

and therefore, $\lim_{r\to+\infty} \varepsilon_2(r) = 0$. Since $\alpha_+ = 1$, we deduce from (B.2.11):

$$\left(\int v\,\gamma_+(\mathrm{d}v) \right) \left(1 - \int_{-r}^{0} \gamma_-(\mathrm{d}w) \right) = \varepsilon_1(r) + \varepsilon_2(r) \ .$$

Since μ is not the Dirac measure at 0, then $\gamma_+(]0,+\infty[) > 0$. Therefore, letting r tend to ∞, we obtain $\alpha_- = 1$, which contradicts the assumption $\alpha_- < 1$. Thus, $\alpha_- = 1$ and $\alpha_+ = 1$.

5. We now prove point ii). We have already seen: $\gamma_+ \ge \mu_+$ and $\gamma_- \ge \mu_-$. The property: $\gamma_+ \ll \mu_+$ follows directly from (B.2.3). More precisely, the Radon-Nikodym density of γ_+ with respect to μ_+ is given by:

$$\left(\int_{\mathbb{R}_-^*} \frac{-w}{v-w}\,\gamma_-(\mathrm{d}w) \right)^{-1} ,$$

which is well defined since γ_- is a probability and $\dfrac{-w}{v-w}$ is > 0 for $w < 0$ and $v \ge 0$. On the other hand, since μ is not the Dirac measure at 0, then $\gamma_+(]0,+\infty[) > 0$. By (B.2.4), this easily entails the property: $\gamma_- \ll \mu_-$, the Radon-Nikodym density of γ_- with respect to μ_- being given by:

$$\left(\int_{\mathbb{R}_+} \frac{v}{v-w}\,\gamma_+(\mathrm{d}v) \right)^{-1} .$$

On the other hand, we have for $v \ge 0$ and $w < 0$,

$$\frac{1}{2}(v \wedge (-w)) \le \frac{-vw}{v-w} \le v \wedge (-w) \tag{B.2.12}$$

Moreover, we deduce from (B.2.3) and (B.2.4)

$$\mathbb{E}\left[|X|^p\right] = \int\!\!\int \frac{-vw}{v-w} \left(v^{p-1} + (-w)^{p-1} \right) \gamma_+(dv)\gamma_-(dw) \tag{B.2.13}$$

for every $p \ge 1$. Then, (B.2.1) and (B.2.2) in Theorem B.2.1 follow directly from (B.2.12) and (B.2.13).

$\square$

We have obtained a theorem of existence, thanks to the application of the Schauder-Tychonoff fixed point theorem, which, of course, says nothing about the uniqueness of the pair (γ_+, γ_-) of probabilities satisfying the conditions $(B.2.3)$ and $(B.2.4)$. However, the following theorem states that this uniqueness holds.

Theorem B.2.3. *Assume $\mu \neq \delta_0$. Then the laws of the r.v.'s V and W satisfying Point i) in Theorem B.2.1 are uniquely determined by μ.*

Proof of Theorem B.2.3

Consider $\left(\gamma_+^{(j)}, \gamma_-^{(j)}\right)$, $j = 1, 2$, two pairs of probabilities in $\mathcal{M}_+ \times \mathcal{M}_+$ satisfying (B.2.3) and (B.2.4). We set, for $j = 1, 2$, $v \geq 0$ and $w < 0$,

$$a^{(j)}(v) = \int_{\mathbb{R}_-^*} \frac{-w}{v - w} \gamma_-^{(j)}(dw), \tag{B.2.14}$$

$$b^{(j)}(w) = \int_{\mathbb{R}_+} \frac{v}{v - w} \gamma_+^{(j)}(dv). \tag{B.2.15}$$

By (B.2.3) and (B.2.4), we have:

$$\gamma_+^{(j)} = \frac{1}{a^{(j)}} \mu_+ \quad \text{and} \quad \gamma_-^{(j)} = \frac{1}{b^{(j)}} \mu_- \tag{B.2.16}$$

On the other hand, the following obvious equality holds:

$$\int\int_{\mathbb{R}_+ \times \mathbb{R}_-^*} \frac{v - w}{v - w} \left(\gamma_+^{(1)}(dv) + \gamma_+^{(2)}(dv)\right) \left(\gamma_-^{(1)}(dw) + \gamma_-^{(2)}(dw)\right) = 4 \tag{B.2.17}$$

Therefore, developing (B.2.17) and using (B.2.14), (B.2.15) and (B.2.16), we obtain:

$$\int_{\mathbb{R}^+} \left(a^{(1)}(v) + a^{(2)}(v)\right) \left(\frac{1}{a^{(1)}(v)} + \frac{1}{a^{(2)}(v)}\right) \mu_+(dv) + \int_{\mathbb{R}_-^*} \left(b^{(1)}(w) + b^{(2)}(w)\right) \left(\frac{1}{b^{(1)}(w)} + \frac{1}{b^{(2)}(w)}\right) \mu_-(dw) = 4 \tag{B.2.18}$$

Now, for $x > 0$, $x + \dfrac{1}{x} \geq 2$, and $x + \dfrac{1}{x} = 2$ if and only if $x = 1$. Therefore,

$$\left(a^{(1)}(v) + a^{(2)}(v)\right) \left(\frac{1}{a^{(1)}(v)} + \frac{1}{a^{(2)}(v)}\right) \geq 4$$

and $\left(a^{(1)}(v) + a^{(2)}(v)\right) \left(\dfrac{1}{a^{(1)}(v)} + \dfrac{1}{a^{(2)}(v)}\right) = 4$ if and only if $a^{(1)}(v) = a^{(2)}(v)$, and similarly with $b^{(1)}(w)$ and $b^{(2)}(w)$. Since μ is a probability, we deduce from (B.2.18) and the preceding that:

$$a^{(1)}(v) = a^{(2)}(v) \ \ \mu_+\text{-a.s.} \quad \text{and} \quad b^{(1)}(w) = b^{(2)}(w) \ \ \mu_-\text{-a.s.}$$

We then deduce from (B.2.16):

$$\gamma_+^{(1)} = \gamma_+^{(2)} \quad \text{and} \quad \gamma_-^{(1)} = \gamma_-^{(2)},$$

which is the desired result.

$\square$.

B.2.2.1 Remark

We have:

$$\forall v \geq 0,\ \forall w < 0, \qquad \frac{-w}{v-w} \geq \frac{1}{(v \vee 1)} \frac{-w}{1-w} \ .$$

Therefore, by (B.2.3), for $p > 1$:

$$\mathbb{E}[V^{p-1}] \leq \left(\int \frac{-w}{1-w} \, \gamma_-(\mathrm{d}w) \right)^{-1} \int (v \vee 1) \, v^{p-1} \, \mu_+(\mathrm{d}v) \ ,$$

and similarly for $\mathbb{E}[(-W)^{p-1}]$. Consequently,

$$\mathbb{E}[|X|^p] < \infty \Longrightarrow \mathbb{E}[V^{p-1}] < \infty \quad \text{and} \quad \mathbb{E}[(-W)^{p-1}] < \infty \ .$$

However, the converse generally does not hold (see Example B.2.3.3 below), but it holds if $p \geq 2$ (see Remark B.3.0.7).

B.2.2.2 Remark

If we no longer require the independence of the two r.v.'s V and W, then, easy computations show that Theorem B.2.1 is still satisfied upon taking for the law of the couple (V, W):

$$2 \, (\mathbb{E}[|X|])^{-1} \, (v - w) \, \mathrm{d}\mu_+(v) \mathrm{d}\mu_-(w). \tag{B.2.19}$$

This explicit formula, which results at once from [Bre68, 13.3, Problem 2], appears in [Hal68]. The results stated in the following Sections B.3 and B.4 remain valid with the law of the couple (V, W) given by (B.2.19), except that, in Theorem B.3.2, one must take care of replacing $\mathbb{E}[V]\mathbb{E}[-W]$ by $\mathbb{E}[-VW]$. Thus the difference between our embedding method and the one which relies on the Breiman-Hall formula is that we impose the independence of V and W. We then have the uniqueness of the laws of V and W (Theorem B.2.3) but no general explicit formula.

B.2.3 Some examples

In this subsection, we develop some explicit examples. We keep the previous notation. For $x \in \mathbb{R}$, δ_x denotes the Dirac measure at x.

B.2.3.1

Let $0 < \alpha < 1$ and $x > 0$. We define $\mu_+ = \alpha\, \delta_x$ and we take for μ_- any measure in $\mathcal{M}_-$ such that

$$\int \mu_-(\mathrm{d}w) = 1 - \alpha \qquad \text{and} \qquad \int w\, \mu_-(\mathrm{d}w) = -\alpha\, x\ .$$

Then, the unique pair of probabilities (γ_+, γ_-) satisfying (B.2.3) and (B.2.4) is given by:

$$\gamma_+ = \delta_x \qquad \text{and} \qquad \gamma_-(\mathrm{d}w) = \left(1 - \frac{w}{x}\right)\, \mu_-(\mathrm{d}w)\ .$$

B.2.3.2

Let $0 < \alpha < 1$ and $0 < x < y$. We consider a symmetric measure μ such that:

$$\mu_+ = \frac{1}{2}\, (\alpha\, \delta_x + (1 - \alpha)\, \delta_y)\ .$$

By an easy computation, we obtain that the unique pair of probabilities (γ_+, γ_-) satisfying (B.2.3) and (B.2.4) is given by:

$$\gamma_+ = \frac{y - \sqrt{(1-\alpha)\, y^2 + \alpha\, x^2}}{y - x}\, \delta_x + \frac{-x + \sqrt{(1-\alpha)\, y^2 + \alpha\, x^2}}{y - x}\, \delta_y$$

and $\gamma_-(\mathrm{d}w) = \gamma_+(-\mathrm{d}w)$.

B.2.3.3

Let $0 < \alpha < 1$ and $0 < \beta < 1$ such that $\alpha + \beta > 1$. We define μ by:

$$\mu_+(\mathrm{d}v) = \frac{\sin \alpha\pi}{\pi}\, \frac{v^{\alpha-1}}{(1 + v^\beta)\, (1 + 2v^\alpha \cos \alpha\pi + v^{2\alpha})}\, \mathrm{d}v$$

and

$$\mu_-(\mathrm{d}w) = \frac{\sin \beta\pi}{\pi}\, \frac{(-w)^{\beta-1}}{(1 + (-w)^\alpha)\, (1 + 2(-w)^\beta \cos \beta\pi + (-w)^{2\beta})}\, \mathrm{d}w\ .$$

Then, the unique pair of probabilities (γ_+, γ_-) satisfying (B.2.3) and (B.2.4) is given by:

$$\gamma_+(\mathrm{d}v) = \frac{\sin \alpha\pi}{\pi}\, \frac{v^{\alpha-1}}{1 + 2v^\alpha \cos \alpha\pi + v^{2\alpha}}\, \mathrm{d}v = (1 + v^\beta)\mu_+(dv)$$

and

$$\gamma_-(\mathrm{d}w) = \frac{\sin \beta\pi}{\pi}\, \frac{(-w)^{\beta-1}}{1 + 2(-w)^\beta \cos \beta\pi + (-w)^{2\beta}}\, \mathrm{d}w = (1 + (-w)^\alpha)\mu_-(dw)\ .$$

This follows from the classical formula, which gives the Laplace transform of the resolvent of index 1 of a stable subordinator of index α (see Chaumont-Yor [CY03, Exercise 4.2.1]):

$$\frac{1}{1+v^{\alpha}} = \frac{\sin \alpha\pi}{\pi} \int_0^{+\infty} \frac{w^{\alpha}}{(v+w)\,(1+2w^{\alpha}\cos\alpha\pi + w^{2\alpha})} \,\mathrm{d}w \;.$$

We note that, in this example, if $p>1$, the condition: $\mathbb{E}[|X|^p] < \infty$ is satisfied if and only if $p < \alpha+\beta$, whereas the conditions: $\mathbb{E}[V^{p-1}] < \infty$ and $\mathbb{E}[(-W)^{p-1}] < \infty$ are satisfied if and only if $p < 1 + \alpha \wedge \beta$. Now, $\alpha + \beta < 1 + \alpha\wedge\beta$ since $\alpha \vee \beta < 1$. This illustrates Remark B.2.2.1.

B.2.3.4

We now consider a symmetric measure μ such that:

$$\mu_+(\mathrm{d}v) = \frac{2}{\pi}\,(1+v^2)^{-2}\,(1+\frac{2}{\pi}\,v\,\log v)\,\mathrm{d}v \;.$$

By an easy computation, we obtain that the unique pair of probabilities (γ_+, γ_-) satisfying (B.2.3) and (B.2.4) is given by:

$$\gamma_+(dv) = \frac{2}{\pi}\,(1+v^2)^{-1}\,\mathrm{d}v$$

and $\gamma_-(\mathrm{d}w) = \gamma_+(-\mathrm{d}w)$.

B.2.3.5

Let μ be a symmetric measure such that:

$$\mu_+(\mathrm{d}v) = \frac{1}{\pi}\left(\frac{1}{\sqrt{v\,(1-v)}} - \frac{1}{\sqrt{1-v^2}}\right) 1_{]0,1[}(v)\,\mathrm{d}v \;.$$

Then, the unique pair of probabilities (γ_+, γ_-) satisfying (B.2.3) and (B.2.4) is given by:

$$\gamma_+(dv) = \frac{1}{\pi}\,\frac{1}{\sqrt{v\,(1-v)}}\,1_{]0,1[}(v)\,\mathrm{d}v$$

and $\gamma_-(\mathrm{d}w) = \gamma_+(-\mathrm{d}w)$. Thus, γ_+ is the Arcsine law.
This follows from the formula:

$$\frac{1}{\pi}\int_0^1 \frac{w}{v+w}\,\frac{1}{\sqrt{w\,(1-w)}}\,\mathrm{d}w = 1 - \sqrt{\frac{v}{1+v}}\;,$$

which can be found in [BFRY06, (1.18) and (1.23)].

B.3 Uniform integrability

In this section, we consider again an integrable, centered, real-valued r.v. X, and we keep the notation of Theorem B.2.1. We shall study the properties of uniform integrability of the martingale: $B^\tau := (B_{u\wedge\tau}, u \geq 0)$.

Theorem B.3.1. *The martingale B^τ is uniformly integrable. Moreover, if $\mathbb{E}[\phi(X)] < \infty$ where $\phi : \mathbb{R} \to \mathbb{R}_+$ is defined by $\phi(x) = |x| \log^+(|x|)$, then, the martingale B^τ belongs to H^1, i.e.* $\mathbb{E}\left[\sup_{u\geq 0} |B_u^\tau|\right] < \infty$.

Proof of Theorem B.3.1

1. We first prove that B^τ is bounded in L^1. We denote by $\mathbb{E}_{W,V}$ the expectation with respect to the law of (W, V), and by $\mathbb{E}_B$ the expectation with respect to the law of Brownian motion B.

$$\begin{aligned}
\sup_{u\geq 0} \mathbb{E}\left[|B_u^\tau|\right] &= \lim_{u\to+\infty} \uparrow \mathbb{E}\left[|B_u^\tau|\right] \\
&= \lim_{u\to+\infty} \uparrow \mathbb{E}_{W,V}\left[\mathbb{E}_B\left[|B_{u\wedge T_{W,V}}|\right]\right] \\
&= \mathbb{E}_{W,V}\left[\lim_{u\to+\infty} \uparrow \mathbb{E}_B\left[|B_{u\wedge T_{W,V}}|\right]\right] \\
&= \mathbb{E}_{W,V}\left[\mathbb{E}_B\left[|B_{T_{W,V}}|\right]\right] \\
&\ \text{(by the dominated convergence theorem,} \\
&\ \text{since } |B_{u\wedge T_{W,V}}| \leq V \vee (-W)) \\
&= \mathbb{E}\left[|B_\tau|\right] = \mathbb{E}[|X|].
\end{aligned}$$

2. We have:

$$\lambda \mathbb{P}\left(\sup_{u\geq 0} |B_u^\tau| \geq \lambda\right) = \mathbb{E}_{W,V}\left[\lambda \mathbb{P}_B\left(\sup_{u\geq 0} |B_{u\wedge T_{W,V}}| \geq \lambda\right)\right]. \qquad \text{(B.3.1)}$$

Now, since $\sup_{u\geq 0} |B_{u\wedge T_{W,V}}| \leq V \vee (-W)$,

$$\lambda \mathbb{P}_B\left(\sup_{u\geq 0} |B_{u\wedge T_{W,V}}| \geq \lambda\right) \xrightarrow[\lambda\to+\infty]{} 0,$$

and from Doob's maximal inequality and Point 1.:

$$\lambda \mathbb{P}_B\left(\sup_{u\geq 0} |B_{u\wedge T_{W,V}}| \geq \lambda\right) \leq \sup_{u\geq 0} \mathbb{E}_B\left[|B_{u\wedge T_{W,V}}|\right] = \mathbb{E}_B\left[|B_{T_{W,V}}|\right],$$

which is $\mathbb{P}_{W,V}$ integrable. Therefore, applying the dominated convergence theorem to the right hand side of (B.3.1), we obtain:

$$\lambda \mathbb{P}\left(\sup_{u\geq 0} |B_u^\tau| \geq \lambda\right) \xrightarrow[\lambda\to+\infty]{} 0.$$

Since B^τ is bounded in L^1, this proves, from Azéma-Gundy-Yor [AGY80, Théorème 1], the uniform integrability of B^τ.

3. We now suppose that $\mathbb{E}[\phi(X)] < \infty$. Applying the previous computation of Point 1. to the submartingale $(\phi(B_u^\tau), u \geq 0)$ (ϕ is convex), we obtain

$$\begin{aligned}\sup_{u\geq 0} \mathbb{E}\left[\phi(B_u^\tau)\right] = \lim_{u\to+\infty} \uparrow \mathbb{E}\left[\phi(B_u^\tau)\right] &= \mathbb{E}\left[\phi(B_\tau)\right] \\ &= \mathbb{E}\left[\phi(X)\right] < \infty. \qquad \text{(B.3.2)}\end{aligned}$$

Note that, under the hypothesis $\mathbb{E}\left[\phi(X)\right] < \infty$, (B.3.2) gives another proof of the fact that B^τ is a uniformly integrable martingale ([Mey66, Chapitre 2, Théorème T22]).
On the other hand, from Doob's $L \log L$ inequality [RY99, p.55],

$$\mathbb{E}\left[\sup_{u\geq 0} |B_u^\tau|\right] \leq \frac{e}{e-1}\left(1 + \sup_{u\geq 0} \mathbb{E}\left[\phi(B_u^\tau)\right]\right) = \frac{e}{e-1}\left(1 + \mathbb{E}\left[\phi(X)\right]\right) < \infty$$

from (B.3.2). Therefore, B^τ belongs to H^1. Actually, the martingale B^τ belongs to the $L \log L$ class (cf. [RY99, Exercise 1.16]).

□

B.3.0.6 Remark

By Azéma-Gundy-Yor [AGY80, Théorème 1], we also deduce from the above Points 1. and 2. that:

$$\lim_{\lambda\to+\infty} \lambda \mathbb{P}\left(\sqrt{\tau} \geq \lambda\right) = 0.$$

We now complete Theorem B.3.1 when the r.v. X admits moments of order $p > 1$. We start with $p = 2$.

Theorem B.3.2. *The following properties are equivalent:*

i) $\mathbb{E}[V] < \infty$ *and* $\mathbb{E}[-W] < \infty$.

ii) $\mathbb{E}[X^2] < \infty$.

iii) $\mathbb{E}[\tau] < \infty$.

iv) The martingale B^τ *is in* H^2.

Moreover, if these properties are satisfied, then

$$\mathbb{E}[X^2] = \mathbb{E}[V]\,\mathbb{E}[-W] \ = \mathbb{E}[\tau].$$

Proof of Theorem B.3.2
We deduce from (B.2.3) and (B.2.4) by addition:

$$\mathbb{E}[X^2] = \mathbb{E}[V]\,\mathbb{E}[-W]\ .$$

This entails the equivalence of properties $i)$ and $ii)$.
On the other hand, if $b \geq 0$ and $a < 0$, the martingale $\left(B^2_{u\wedge T_{a,b}} - (u \wedge T_{a,b}), u \geq 0\right)$ is uniformly integrable and hence, $\mathbb{E}[T_{a,b}] = \mathbb{E}\left[B^2_{T_{a,b}}\right] = -ab$. Consequently,

$$\mathbb{E}[\tau] = \mathbb{E}[T_{W,V}] = -\mathbb{E}[WV] = \mathbb{E}[V]\mathbb{E}[-W].$$

This shows that properties $i)$ and $iii)$ are equivalent.

By Doob's L^2 inequality,

$$\mathbb{E}\left[\left(\sup_{u\geq 0}|B^\tau_u|\right)^2\right] \leq 4\sup_{u\geq 0}\mathbb{E}\left[(B^\tau_u)^2\right] = 4\mathbb{E}[\tau]$$

Hence, $iii) \Longrightarrow iv)$. The converse follows from:

$$\mathbb{E}\left[u\wedge\tau\right] = \mathbb{E}\left[(B^\tau_u)^2\right] \leq \mathbb{E}\left[\left(\sup_{u\geq 0}|B^\tau_u|\right)^2\right],$$

upon letting u tend to $+\infty$. Therefore:

$$\mathbb{E}[\tau] \leq \mathbb{E}\left[\left(\sup_{u\geq 0}|B^\tau_u|\right)^2\right] \leq 4\mathbb{E}[\tau].$$

□

We now replace the L^2 space by L^p for $p > 1$.

Theorem B.3.3. *Let $p > 1$. The following properties are equivalent:*

i) $\mathbb{E}[(V\wedge(-W))(V^{p-1} + (-W)^{p-1})] < \infty$.

ii) $\mathbb{E}[|X|^p] < \infty$.

iii) $\mathbb{E}[\tau^{p/2}] < \infty$.

iv) The martingale B^τ is in H^p.

Proof of Theorem B.3.3

1. By (B.2.2), properties $i)$ and $ii)$ are equivalent.

2. We now prove that *ii)* is equivalent to *iii)*. We fix $p > 1$. Applying Doob's and the Burkholder-Davis-Gundy inequalities to the bounded martingale $\left(B_{u\wedge T_{a,b}}, u \geq 0\right)$, we obtain that there exist constants c and C (depending only on p) such that, for every $a < 0$ and $b \geq 0$,

$$c\mathbb{E}\left[(T_{a,b})^{p/2}\right] \leq \mathbb{E}\left[|B_{T_{a,b}}|^p\right] \leq C\mathbb{E}\left[(T_{a,b})^{p/2}\right].$$

   Hence, we obtain, since $\tau = T_{W,V}$,

$$c\mathbb{E}\left[\tau^{p/2}\right] \leq \mathbb{E}[|X|^p] \leq C\mathbb{E}\left[\tau^{p/2}\right]$$

   which entails *ii)* $\iff$ *iii)*.

3. Finally, *iii)* $\iff$ *iv)* is a direct consequence of the Burkholder-Davis-Gundy inequalities.

□

B.3.0.7 Remark

If $p \geq 2$, the property $\mathbb{E}\left[|X|^p\right] < \infty$ is equivalent to: $\mathbb{E}[V^{p-1}] < \infty$ and $\mathbb{E}[(-W)^{p-1}] < \infty$. This is proven in Theorem B.3.2 for $p = 2$.
Now, suppose $p > 2$. We saw in Remark B.2.2.1 that:

$$\mathbb{E}[|X|^p] < \infty \implies \mathbb{E}[V^{p-1}] < \infty \text{ and } \mathbb{E}[(-W)^{p-1}] < \infty.$$

Conversely, suppose $\mathbb{E}[V^{p-1}] < \infty$ and $\mathbb{E}[(-W)^{p-1}] < \infty$. In particular, $\mathbb{E}[V] < \infty$ and $\mathbb{E}[(-W)] < \infty$. We deduce from (B.2.3) and (B.2.4):

$$\mathbb{E}[|X|^p] \leq \mathbb{E}[-W]\,\mathbb{E}[V^{p-1}] + \mathbb{E}[V]\,\mathbb{E}[(-W)^{p-1}]$$

which entails $\mathbb{E}[|X|^p] < \infty$.

B.4 Construction of self-similar martingales

In this Section, we consider a real valued, centered, random variable X. Let V, W, be as in Theorem B.2.1. We set:

$$\tau_t = \inf\{u \geq 0\,;\ B_u = \sqrt{t}\,V \ \text{ or } \ B_u = \sqrt{t}\,W\}\,.$$

Theorem B.4.1.

i) The process $(B_{\tau_t}\,,\ t \geq 0)$ *is a left-continuous martingale such that, for every fixed* t, $B_{\tau_t} \overset{(law)}{=} \sqrt{t}\,X$.

ii) For any $c>0$,

$$(B_{\tau_{c^2 t}}\ ,\ t\geq 0) \overset{(law)}{=} (c\,B_{\tau_t}\ ,\ t\geq 0)\ .$$

iii) The process $(B_{\tau_t}\ ,\ t\geq 0)$ is an inhomogeneous Markov process.

*In particular, $(B_{\tau_t}, t\geq 0)$ is a martingale associated to the peacock $(\sqrt{t}X, t\geq 0)$ (see **I.4** in the General Introduction).*

Proof of Theorem B.4.1

1. By the definition of times τ_t and the continuity of B, one easily sees that the process $(\tau_t\ ,\ t\geq 0)$ is a left-continuous increasing process. As a consequence, $(B_{\tau_t}\ ,\ t\geq 0)$ is a left-continuous process.

2. Since, for a given $t\geq 0$, $(M_u := B_{u\wedge\tau_t}\ ,\ u\geq 0)$ is a uniformly integrable martingale, and for $s<t$, $\tau_s\leq\tau_t$, then $(B_{\tau_t}, t\geq 0)$ is a martingale.
Let, for $c>0$, $(B_t^{(c)} := c\,B_{c^{-2}t}\ ,t\geq 0)$, and denote by $(\tau_t^{(c)})$ the family of stopping times associated with the Brownian motion $B^{(c)}$. In other words,
$$\tau_t^{(c)} = \inf\{u\geq 0\ ;\ B_u^{(c)} = \sqrt{t}\,V \ \text{ or } \ B_u^{(c)} = \sqrt{t}\,W\}\ .$$
We easily obtain, for every $t\geq 0$, $\tau_t^{(c)} = c^2\,\tau_{c^{-2}t}$ and then, $B^{(c)}_{\tau^{(c)}_{c^2 t}} = c\ B_{\tau_t}$, which proves point iii) since $(B_t^{(c)}\ ,\ t\geq 0) \overset{(\mathrm{law})}{=} (B_t\ ,\ t\geq 0)$. Moreover, since $B_{\tau_1} \overset{(\mathrm{law})}{=} X$, we also have, for every $t\geq 0$, $B_{\tau_t} \overset{(\mathrm{law})}{=} \sqrt{t}\,X$.

3. We now consider the Brownian motion B as a strong Markov process in $\mathbb{R}$. We may define $\widetilde{\tau}_t$ by:
$$\widetilde{\tau}_t = \inf\{u\geq 0\ ;\ B_u \notin]\sqrt{t}\,W, \sqrt{t}\,V[\}\ .$$
(Note that $\widetilde{\tau}_t = \tau_t$ under $\mathbb{P}_0$, whereas, if $x\neq 0$, then $\widetilde{\tau}_t\neq\tau_t$ under $\mathbb{P}_x$.)
For $s<t$, we have with the usual notation about time translation operators (θ_u),
$$\widetilde{\tau}_t = \widetilde{\tau}_s + \widetilde{\tau}_t\circ\theta_{\widetilde{\tau}_s}$$
and consequently: $B_{\widetilde{\tau}_t} = B_{\widetilde{\tau}_t}\circ\theta_{\widetilde{\tau}_s}$, which entails, for f a bounded Borel function,
$$\mathbb{E}[f(B_{\widetilde{\tau}_t})\mid\mathcal{F}_{\widetilde{\tau}_s}] = \mathbb{E}_{B_{\widetilde{\tau}_s}}[f(B_{\widetilde{\tau}_t})]\ ,$$

which proves point ii). More precisely, the transition semi group: $(P_{s,t}\ ,\ 0 \leq s < t)$ is given by:

$$
\begin{aligned}
&P_{s,t}f(x) = \\
&\mathbb{E}\left[\left(f(\sqrt{t}\,V)\,\frac{x-\sqrt{t}\,W}{\sqrt{t}\,(V-W)} + f(\sqrt{t}\,W)\,\frac{-x+\sqrt{t}\,V}{\sqrt{t}\,(V-W)}\right)\ 1_{]\sqrt{t}\,W,\sqrt{t}\,V[}(x)\right] \\
&+f(x)\,\mathbb{P}\left(x \notin]\sqrt{t}\,W, \sqrt{t}\,V[\right)\ .
\end{aligned}
$$

Thus, $(P_{s,t}\ ,\ 0 \leq s < t)$ is a transition semi group of a very special kind since, actually, $P_{s,t}$ does not depend on $s \in [0,t[$.

$\square$

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